The Information Velocity of Packet-Erasure Links

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Joint work with

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IPG Seminar, EPFL

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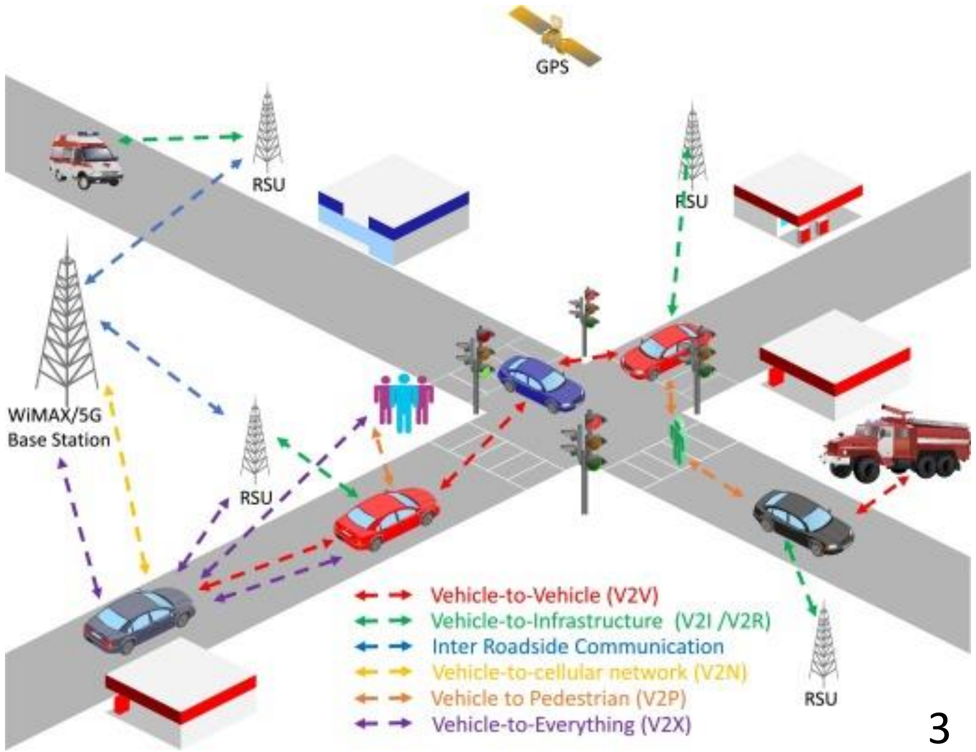
Outline

- Motivation & introduction
- Model: information velocity (IV) & error exponent (EE)
- Main results
- Derivation: IV & EE of a single packet
- Derivation: IV & EE of a stream of packets
- Numerical results
- Summary & outlook

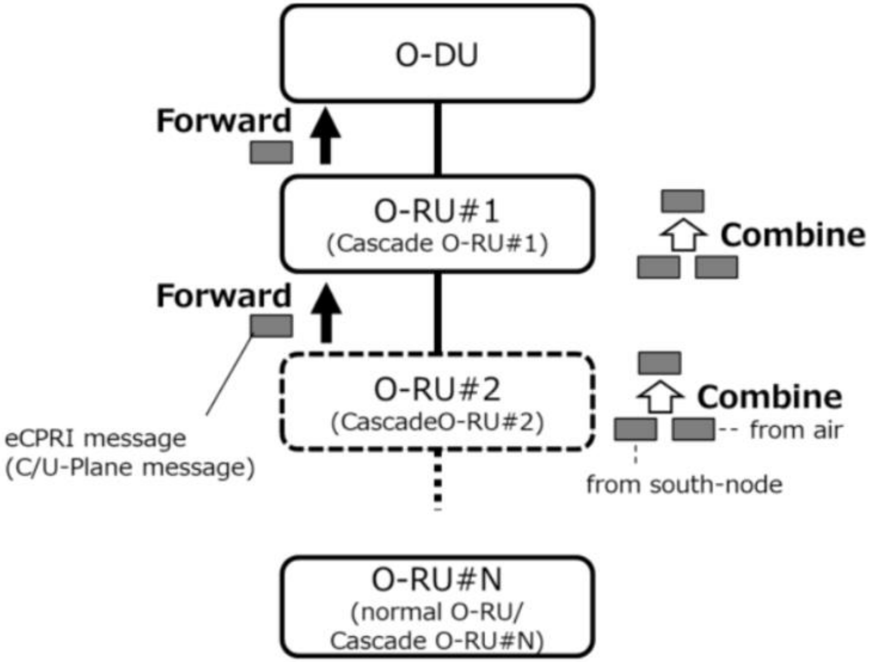
Motivation: 5G

- Huge increase in the number of connected devices
- Device-to-device (D2D) communications ⇒ **relaying**
  - Cellular vehicle-to-everything (C-V2X): Platooning (URLLC)

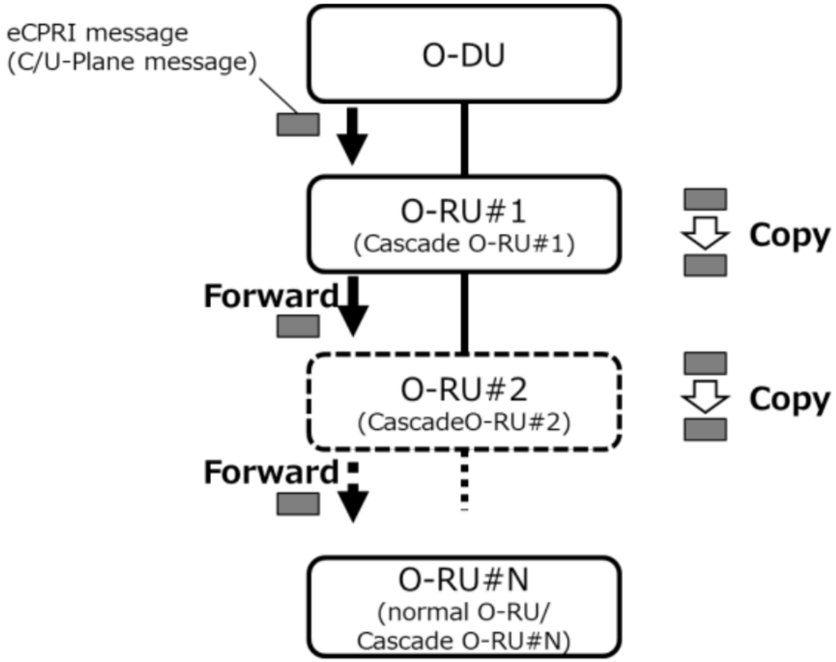


Motivation: 5G

- Huge increase in the number of connected devices
- Device-to-device (D2D) communications ⇒ **relaying**
  - Open Radio Access Network (O-RAN): shared cell, cascade mode



Cascade mode



Cascade mode

Motivation: 5G

- Huge increase in the number of connected devices
- Device-to-device (D2D) communications ⇒ **relaying**

Requires

- highly-reliable packet transmission
- Low latency

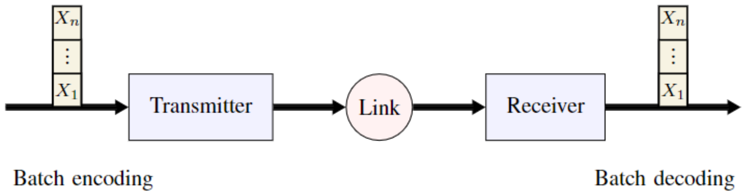
Fundamental limits?

- ***"First order":*** Given **delay**, max **# links** a message can "reliably" traverse
- ***"Second order":*** Error probability exponential decay rate

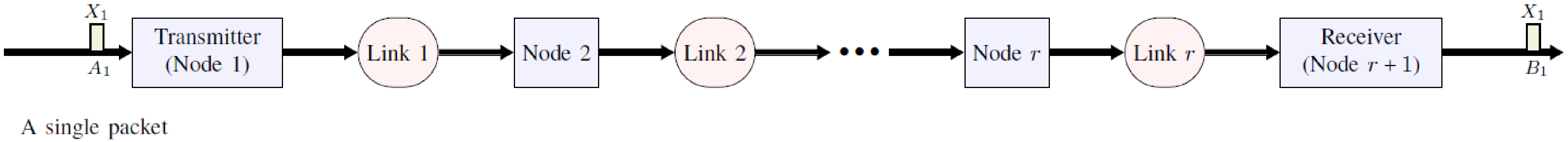
Comparison of the Two Problems

Property	Classical problem	Dual problem
# links	Single	Multiple
# messages	Multiple	Single • Will be extended to multiple messages

Classical:



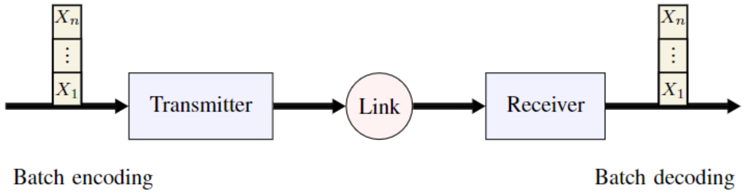
Dual:



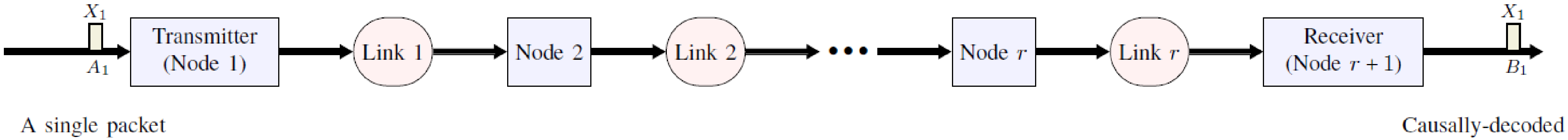
Comparison of the Two Problems

Property	Classical problem	Dual problem
Fundamental limit ("First order")	**Capacity,** C: ✓ • Max rate R of $\left[\frac{\text{\# Messages}}{\text{Time steps}}\right]$ • Error probability $P_e \xrightarrow{\text{Time}\to\infty} 0$	**Information Velocity (IV),** $\vec{V}$: **?** • Max speed α of $\left[\frac{\text{\# Links}}{\text{Time steps}}\right]$ • Error probability $P_e \xrightarrow{\text{Time}\to\infty} 0$
Fundamental limit ("Second order")	**Error Exponent,** E: $@\ R < C$ ✓̸ $E(R) = \lim_{\text{Time}\to\infty} -\frac{1}{\text{Time}} \log P_e$	**Error Exponent (EE),** E: $@\ \alpha < \vec{V}$ **?** $E(\alpha) = \lim_{\text{Time}\to\infty} -\frac{1}{\text{Time}} \log P_e$

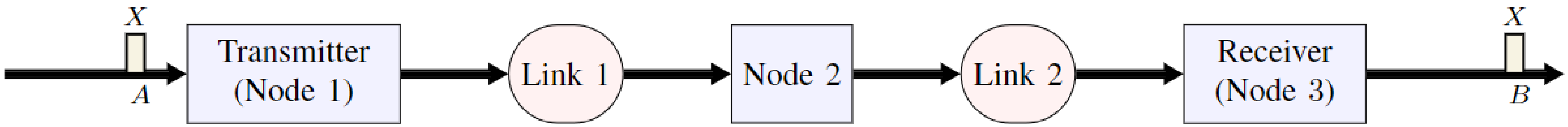
Classical:

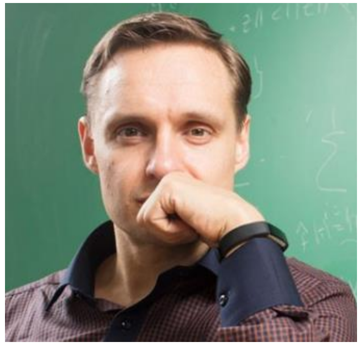


Dual:



Prior Art & Related Work



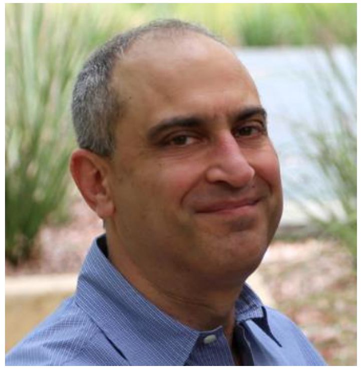


Single bit through two BSCs (bit-flip links):

- [Jog–Loh IT'20][Huleihel–Polyanskiy–Shayevitz ISIT'19]: Bounds on the EE
- [Ling–Scarlett ISIT'21]: EE = EE of single BSC

Finite number of bits through a cascade of BSCs (bit-flip links):

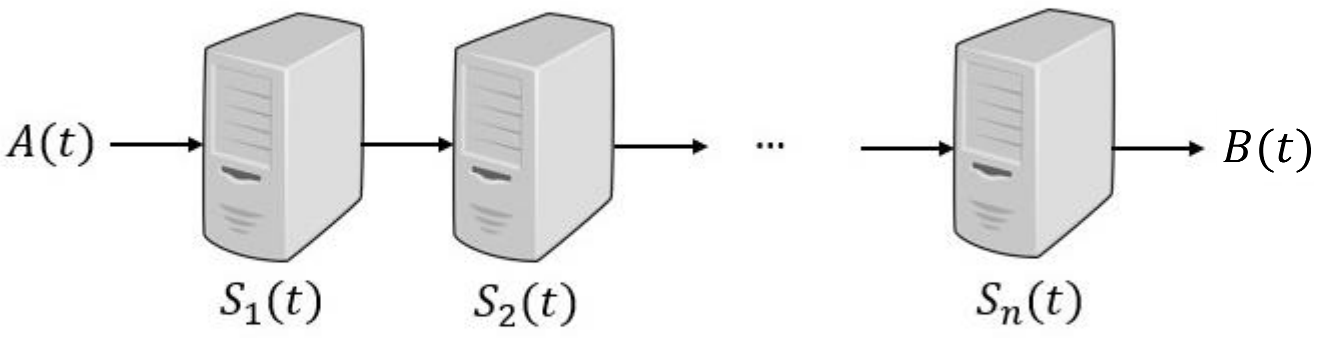
- p probability of bit flip
- [Rajagopalan–Schulman ACM'94]: $\vec{V} \leq 1 - 2p$
  - This result is immediate given the IV for erasure links that we will see in this talk
- [Ling–Scarlett, ArXiv'21]: $0 < \vec{V} \leq (1 - 2p)^2$



What about an online setup?

- Causally, constantly, arriving messages

Prior Art & Related Work



Cascaded computation: Stream of packets through n servers

- Infinite buffers
- Stochastic arrival and service curves
- Stochastic network calculus [Fidler–Rizk Comm surveys ’15]:

  Assuming independent service times, mean E2E delay ∝ # servers

$$\Downarrow$$

$$\vec{V} > 0$$

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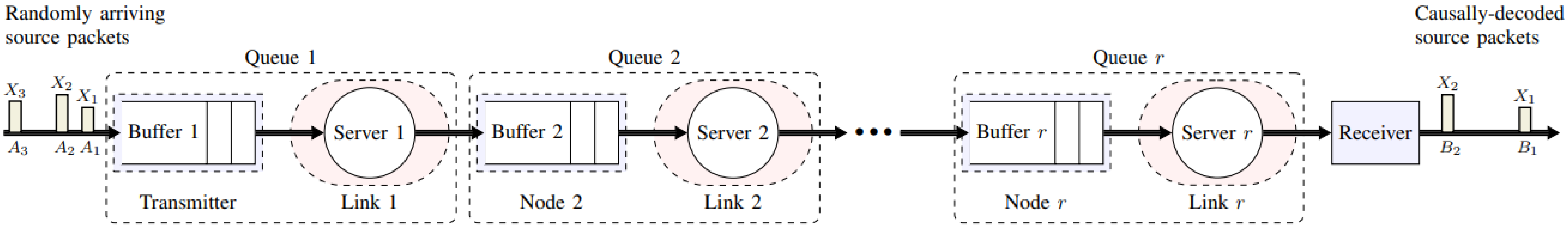

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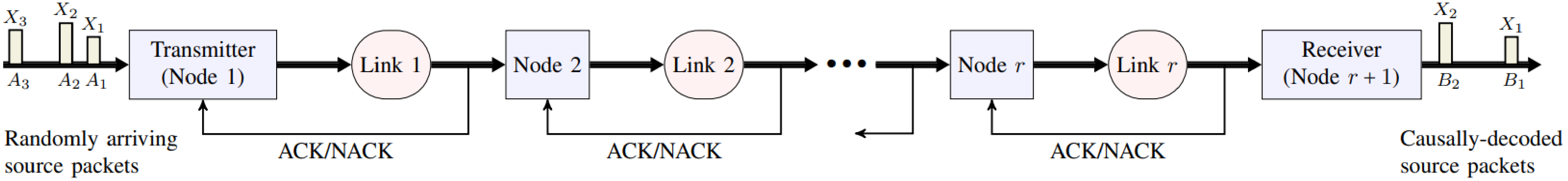
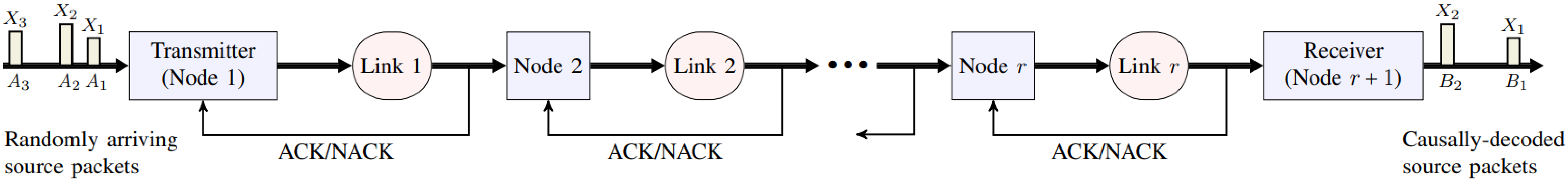
System Model



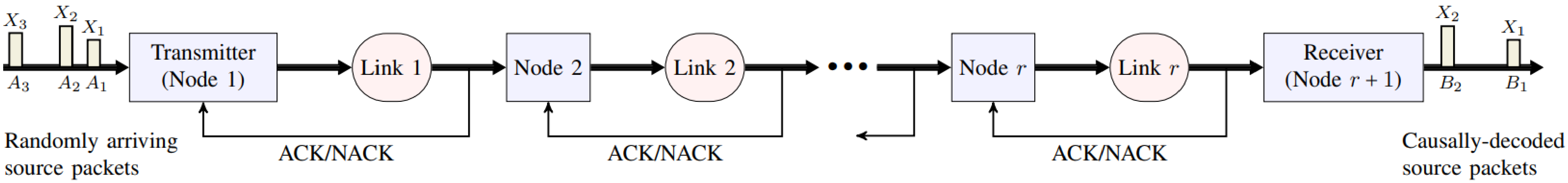
- Cascade of **packet-erasure** links
- IID **Bernoulli erasures** in each link
- Erasures are independent across different links (and times)
- Instantaneous perfect acknowledgment (ACK) feedback
- Stream of causally arriving packets (at random, periodically, …)
- In-order communications

System Model



- Source stream: Packet $m \in \mathbb{Z}$ arrives at time $A_m \in \mathbb{Z}$
- Output of link i serves as the input to node $i+1$
- At each time step: Packet sent in link i is erased with probability p_i
  - Packets are acknowledged upon arrival
  - If packet erased, it is retransmitted until successful arrival at next node
- Departure process
  - Packet $m \in \mathbb{Z}$ arrives at final receiver at time $B_m \in \mathbb{Z}$

System Model



- **Arrive-failure probability:** $P_{\mathrm{e}}(N) \triangleq \sup_{m\in\mathbb{Z}} \Pr(B_m > A_m + N)$

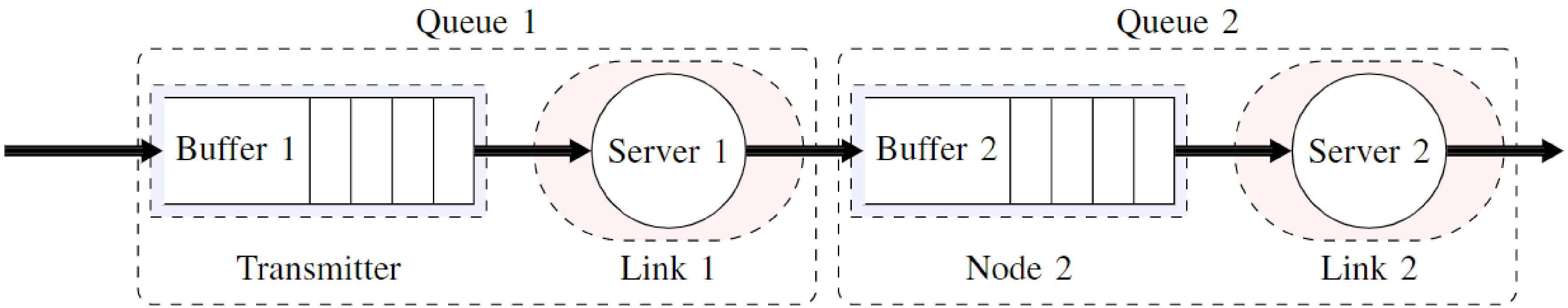
- **Information velocity:** $\vec{V} \triangleq \sup\left\{\alpha > 0 \middle| r = \lceil \alpha N \rceil, \lim_{N\to\infty} P_e(N) = 0\right\}$
  - $r \propto N$ is the proper growth rate

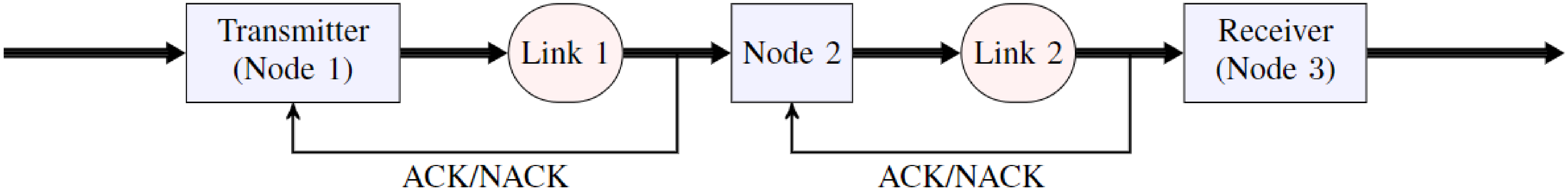
- **Error exponent:** $E \triangleq \lim_{N\to\infty} -\frac{1}{N}\log P_e(N)$ for $\alpha < \vec{V}$

Illustration

Event: $t = 0$: Start

Next: $t = 1$: Packet arrival ($A_1 = 1$)

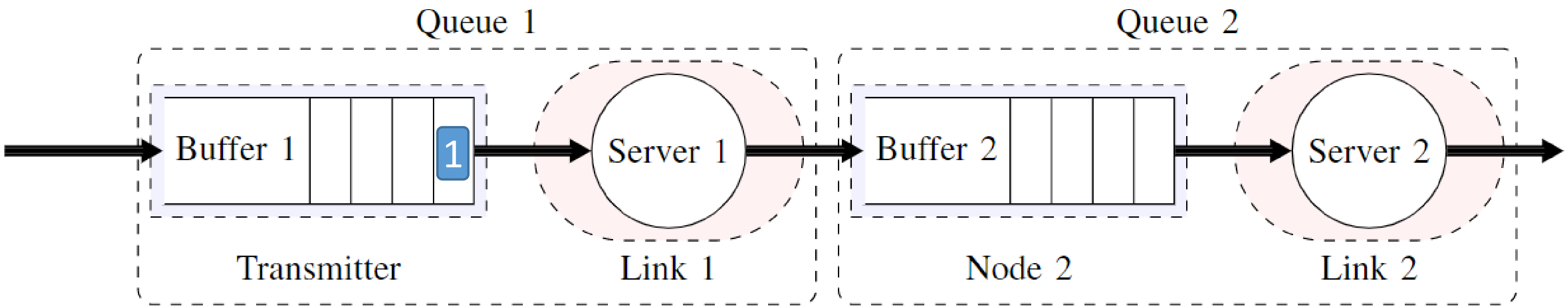


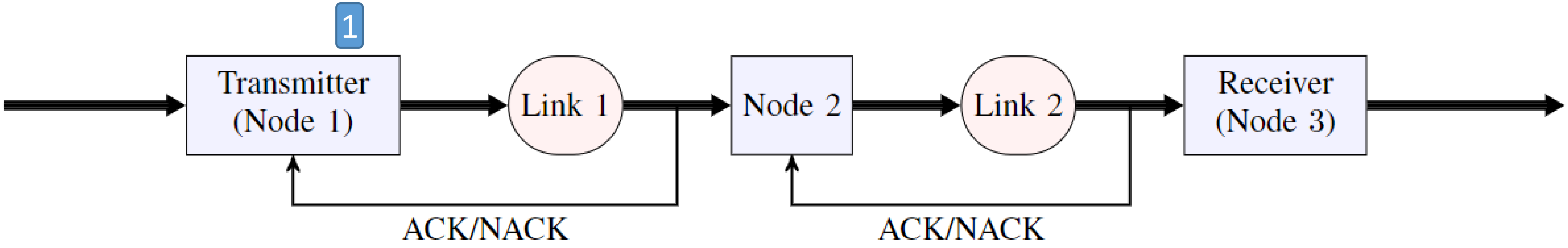


Illustration

Event: $t = 1$: Packet arrival ($A_1 = 1$)

Next: $t = 2$: Packet arrival ($A_2 = 2$) & Link 1 erasure

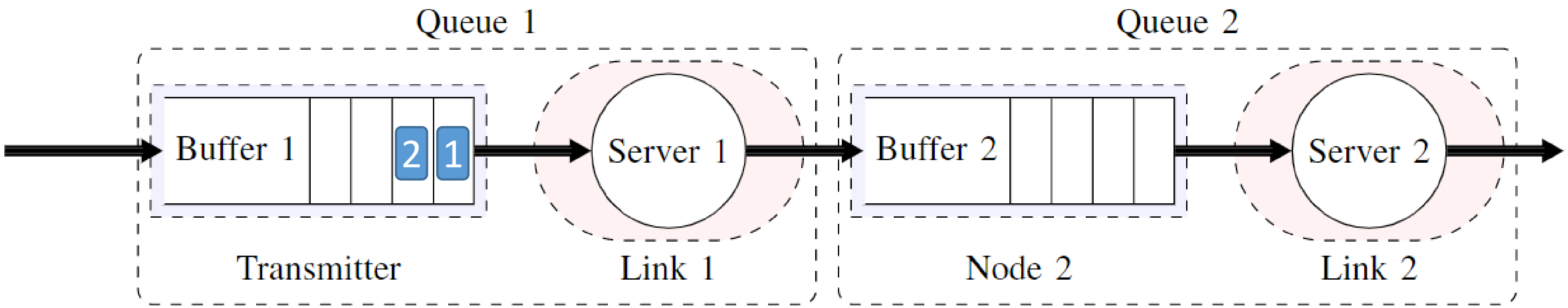


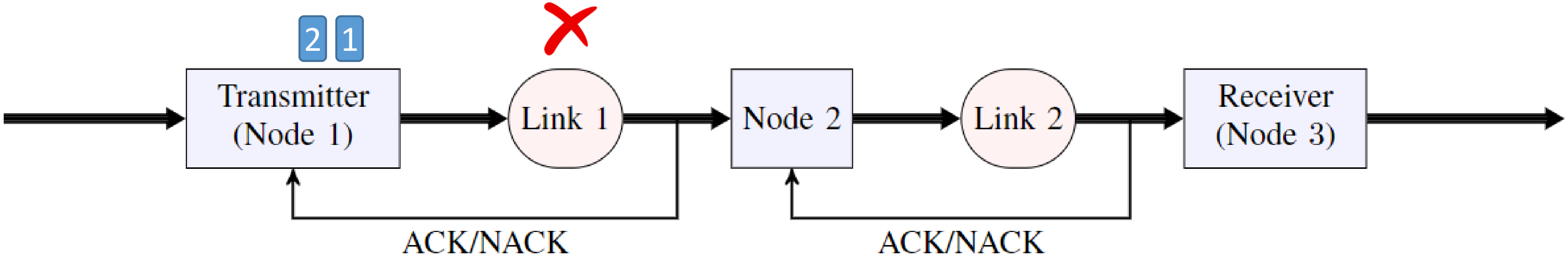


Illustration

Event: $t = 2$: Packet arrival ($A_2 = 2$) & Link 1 erasure

Next: $t = 3$: Success @ Link 1

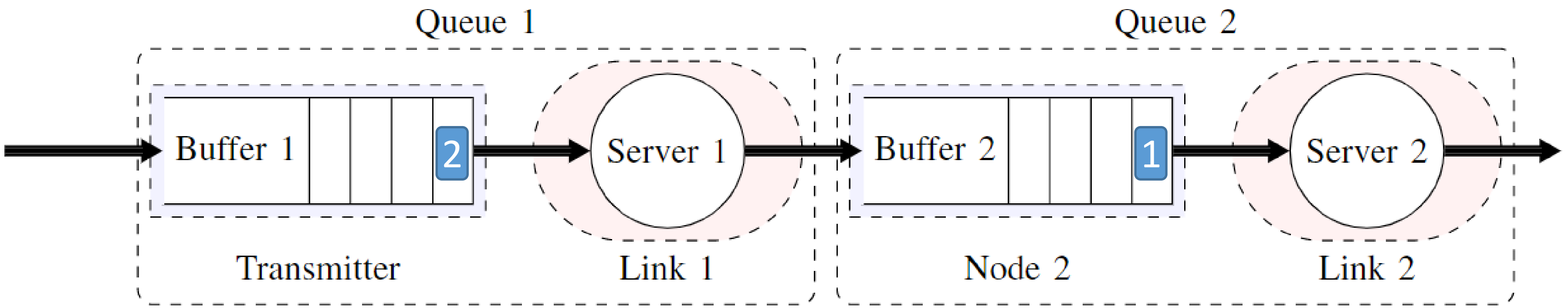


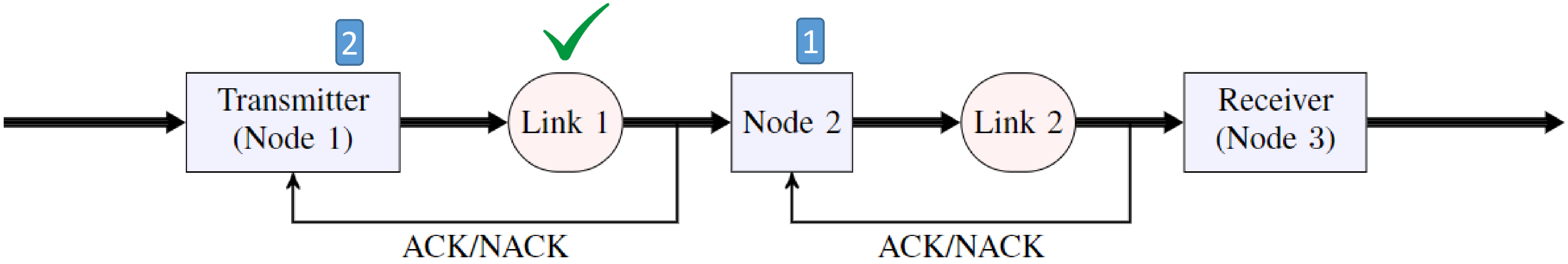


Illustration

Event: $t = 3$: Success @ Link 1

Next: $t = 4$: Failure @ Link 1, success @ Link 2 (B_1 =4) , packet arrival (A_3 =4)

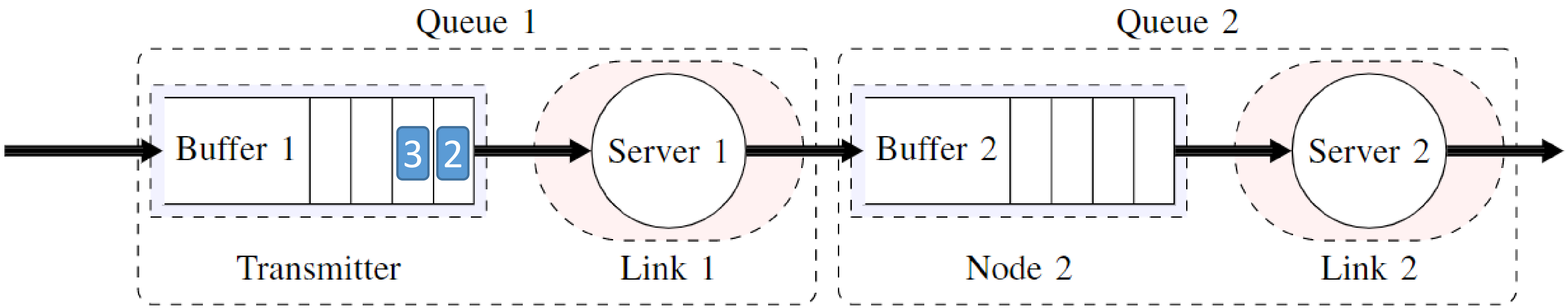


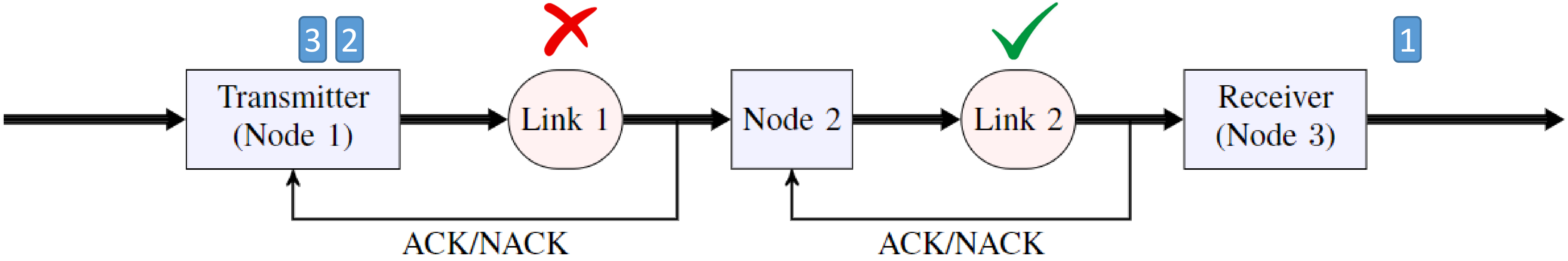


Illustration

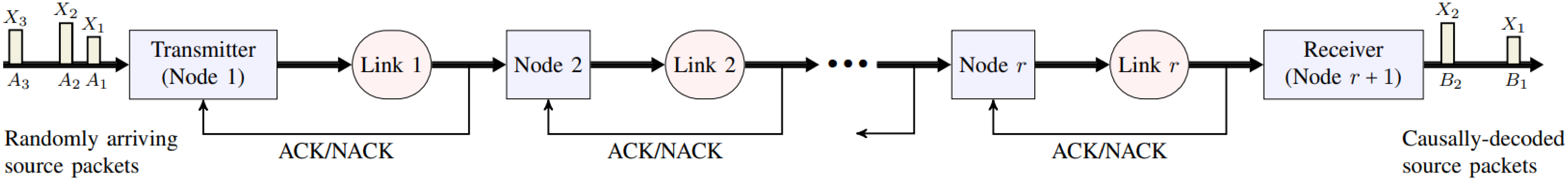
Event: $t = 4$: Failure @ Link 1, success @ Link 2 (B_1 =4) , packet arrival (A_3 =4)

Next:





Main Results



Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **Homogeneous links:** $p_1 = p_2 = \cdots = p_r \triangleq p$
- **IID Bernoulli arrival input process:** IID packet arrivals with probability $\lambda < 1 - p$

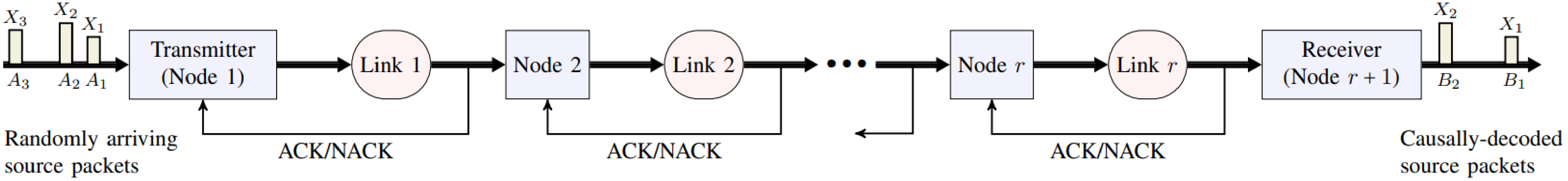
⇒ **IV in steady state:** $\vec{V} = 1 - \frac{p}{1-\lambda}$

⇒ **EE in steady state:** for $\alpha < \vec{V}$: $E = \mathbb{D}\left(\alpha \| 1 - \frac{p}{1-\lambda}\right)$

⇒ For $\lambda \geq 1 - p$: $\vec{V} = 0$

Kullback–Leibler divergence: $\mathbb{D}\,(q \| s) \triangleq q \log \frac{q}{s} + (1-q) \log \frac{1-q}{1-s}$

Main Results



Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

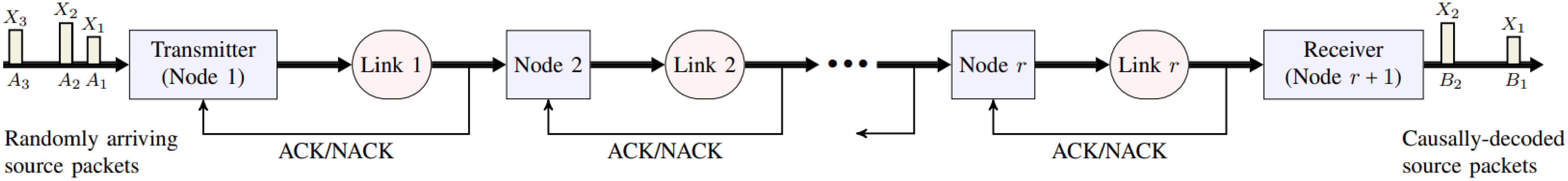
- **Homogeneous links:** $p_1 = p_2 = \cdots = p_r \triangleq p$
- **Stationary ergodic input process** of arrival rate $\lambda < 1 - p$

→ **IV in steady state:** $\vec{V} \overset{=}{\leq} 1 - \frac{p}{1-\lambda}$

→ **EE in steady state:** for $\alpha < \vec{V}$: $E \leq \mathbb{D}\left(\alpha \| 1 - \frac{p}{1-\lambda}\right)$

→ For $\lambda \geq 1 - p$: $\vec{V} = 0$

Main Results



Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

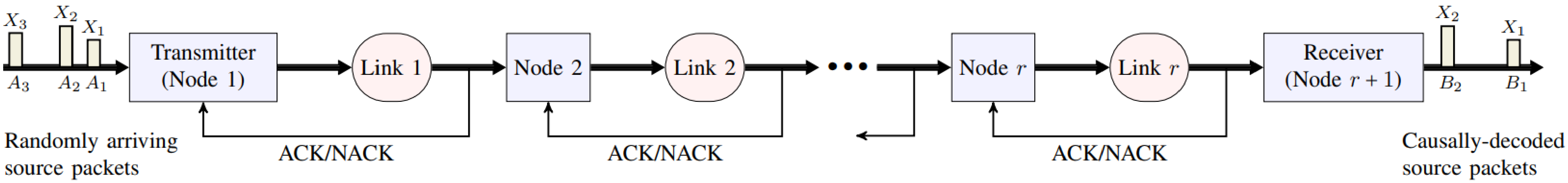
- **Homogeneous links:** $p_1 = p_2 = \cdots = p_r \triangleq p$
- **Deterministic periodic input process:** A packet arrives every $1/\lambda$ time steps
  - More generally, arrivals at times $\lfloor i \cdot a \rfloor$ for a fixed $a \in \mathbb{Q}$ and all $i \in \mathbb{Z}$

→ **IV in steady state:** $\vec{V} \overset{=}{\leq} 1 - \frac{p}{1-\lambda}$

→ **EE in steady state:** for $\alpha < \vec{V}$: $E \leq \mathbb{D}\left(\alpha \| 1 - \frac{p}{1-\lambda}\right)$

→ For $\lambda \geq 1 - p$: $\vec{V} = 0$

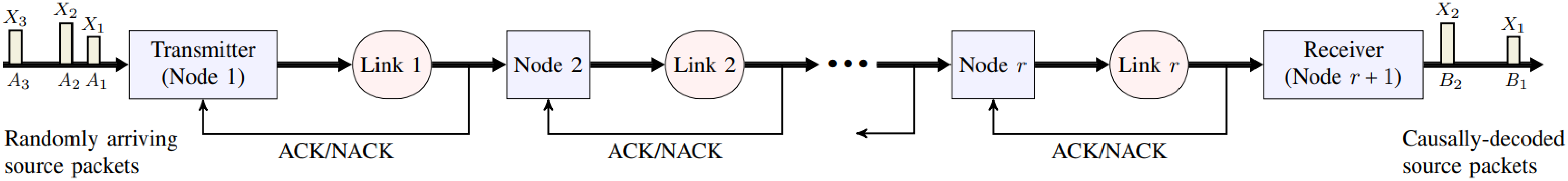
Main Results: Extensions



- Closed-form results for heterogeneous links as well
  - Based on large deviation / method of types techniques
- Treatment can be extended to cyclostationary cycloergodic input processes
  - Cycloergodic processes [Boyles–Gardner IT'83]

and more generally, to arrival process whose long time average → mean rate

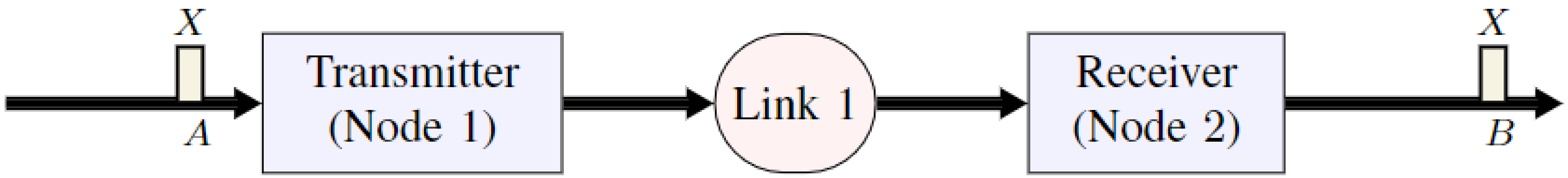
Main Results: Proof Plan



- Single source packet
- Stream of packets for different arrival processes:
  - IID Bernoulli arrival input process—by lifting the results for a single packet
  - Stationary ergodic input process
  - Periodic (deterministic) arrivals

Single Packet

Single Packet, Single Link



- Assume the packet is available at the sender at time $A = 1$
- Single link: $r = 1$

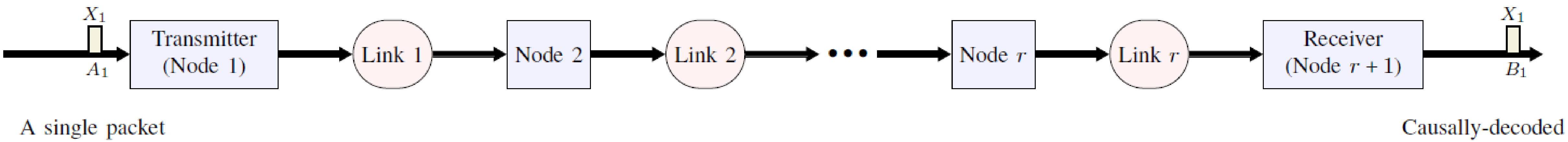
- Arrive-failure probability:

$$P_e(N) = p^N = \exp\{-N \cdot (-\log p)\}$$

$$\Downarrow$$

$$E = -\log p$$

Single Packet, Homogeneous Links



Homogeneous links: $p_1 = p_2 = \cdots = p_r \triangleq p$

Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **IV:** $\vec{V} = 1 - p$
- **EE:** for $\alpha < \vec{V}$: $E = \mathbb{D}(\alpha || 1-p) \triangleq \alpha \log \frac{\alpha}{1-p} + (1-\alpha) \log \frac{1-\alpha}{p}$
- P_e over r links across N time steps is bounded as

$$\frac{(1-p)\sqrt{N}\exp\left\{-N \cdot \mathbb{D}\left(\frac{r-1}{N} \| 1-p\right)\right\}}{\sqrt{8(r-1)(N-r+1)}} \le P_e(N) \le \min\left\{\begin{array}{c} \exp\left\{-(N-1)\cdot \mathbb{D}\left(\frac{r}{N-1}\middle|\middle|1-p\right)\right\}, \\ \frac{(1-p)\sqrt{N}}{\sqrt{2\pi(r-1)(N-r+1)}} \cdot \frac{\exp\left\{-N \cdot \mathbb{D}\left(\frac{r-1}{N} \| 1-p\right)\right\}}{1-\exp\left\{-N \cdot \mathbb{D}\left(\frac{r-1}{N} \| 1-p\right)\right\}} \end{array}\right\}$$

Corollary: For $r = o(N)$, $E = -\log p$

Single Packet, Homogeneous Links

Proof:

- t_i—time of arrival at node $i+1$ (over link i)
- $\tau_i \triangleq t_i - t_{i-1}$—delay caused by link i
- Clearly, $\tau_1, \tau_2, \dots, \tau_r \sim \text{IID}\ \mathcal{Geo}(1-p)$

$$P_e(N) = \Pr\left(\sum_{i=1}^{r} \tau_i > N\right)$$

IV expression derivation: For $\alpha \triangleq r/N$

$$\lim_{N\to\infty} P_e(N) = \lim_{N\to\infty} \Pr\left(\frac{1}{r}\sum_{i=1}^{r} \tau_i > \frac{N}{r}\right) = \begin{cases} 0, & \mathbb{E}[\tau] < 1/\alpha \\ 1, & \mathbb{E}[\tau] > 1/\alpha \end{cases}$$

- $\mathbb{E}[\tau] = \frac{1}{1-p} \gtrless \frac{1}{\alpha} \quad \Rightarrow \quad \vec{V} = 1-p$

Single Packet, Homogeneous Links

EE expression derivation: By Chernoff's upper bound on P_e:

$$P_e(N) = \Pr\left(\sum_{i=1}^{r} \tau_i > N\right) \leq \exp\left\{-\sup_{s>0}\{(N-1)s - \alpha \log M_\tau(s)\}\right\}$$

$$= \exp\left\{-(N-1) \cdot \mathbb{D}\left(\frac{r}{N-1}\middle|\middle|1-p\right)\right\}$$

By Cramér's theorem, the above achievable exponent is tight:

$$E = -\sup_{s>0}\{s - \alpha \log M_\tau(s)\} = \mathbb{D}(\alpha \| 1-p)$$

where $M_\tau(s) \triangleq \mathbb{E}[\exp\{s\tau\}] = \frac{(1-p)\exp\{s\}}{1-p\exp\{s\}}$

Single Packet, Homogeneous Links

Remarks:

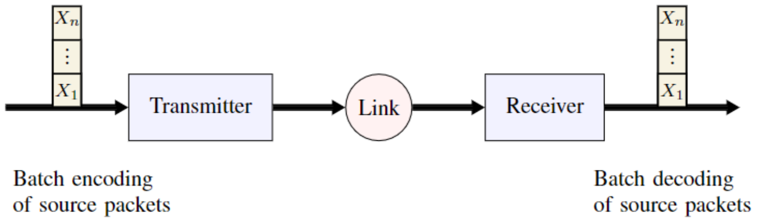
- $1 - p = \vec{V}$—expected # relays the message traverses per time step
- Can be used as alternative def. of IV over erasure links
- Original def. is akin to “almost lossless source coding” whereas alternative def. is akin to “lossless (variable-length) source coding”

1 Packet over r Link $\Leftrightarrow$ r Packets over 1 Link

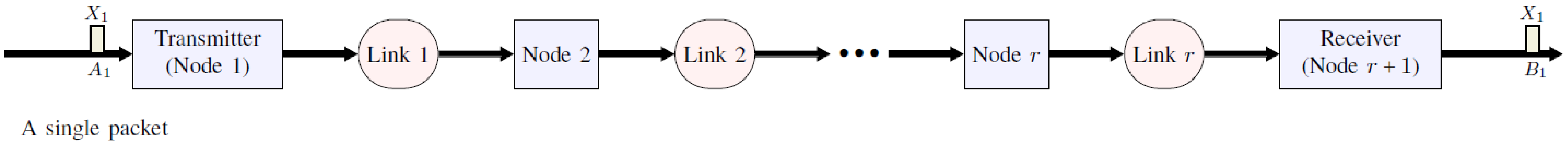
Property	Classical problem	Dual problem
# links	Single	Multiple
# messages	Multiple	Single
Homogeneous erasure links	**Capacity:** $C = 1 - p$	**Information Velocity (IV)** $\vec{V} = 1 - p$

- **Observation for homogeneous links:** $C = \vec{V}$
- **Is it a coincidence?**

Classical:



Dual:



1 Packet over r Link ⇔ r Packets over 1 Link

Property	Classical problem	Dual problem
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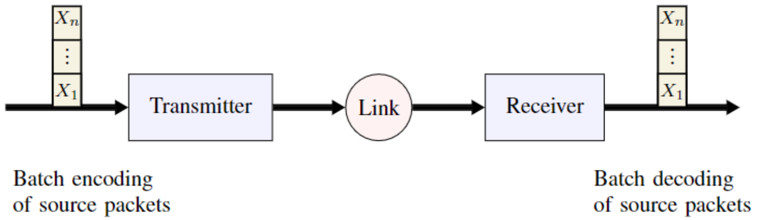
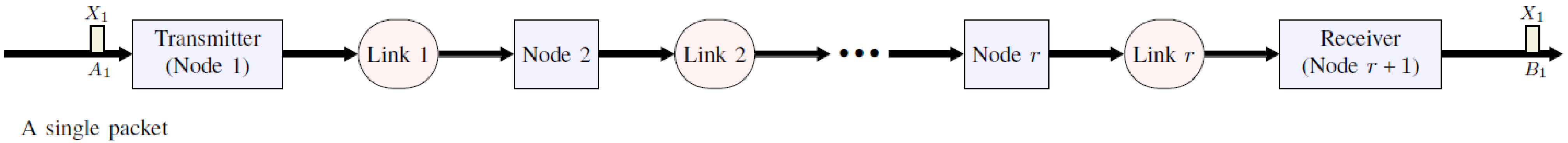
Homogeneous erasure links	**Capacity:** $C = 1 - p$	**Information Velocity (IV)** $\vec{V} = 1 - p$

- **Observation for homogeneous links:** $C = \vec{V}$
- **Is it a coincidence?** Of course not. ☺

Classical:

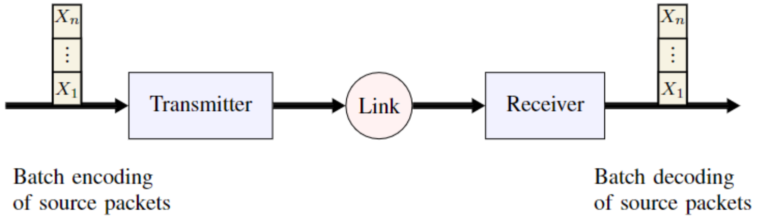


Dual:



1 Packet over r Link $\Leftrightarrow$ r Packets over 1 Link

Classical:



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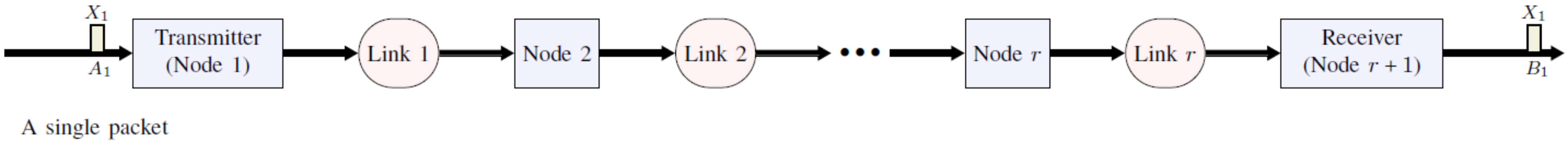


Homogeneous erasure links:

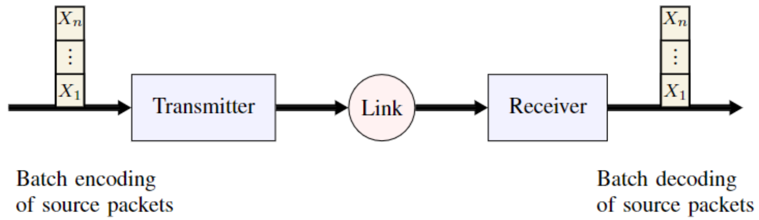
- The two problems are in fact equivalent for erasure links
- More generally, for erasure links with ACK feedback:

m messages over r links $\Leftrightarrow$ r messages over m links

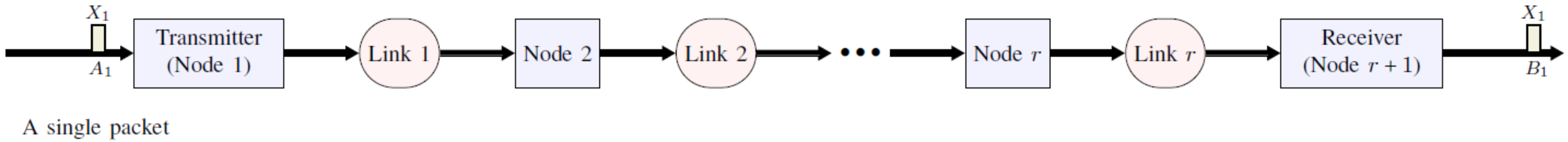
Remark: For single packet transmission, the same results hold without feedback

1 Packet over r Link ⇔ r Packets over 1 Link

Classical:



Dual:



Heterogeneous erasure links: The parallel "classical problem" is weird:

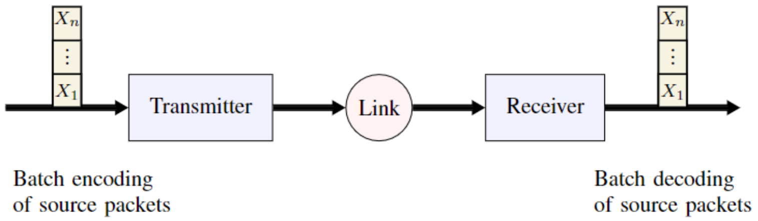
- Upon a successful packet-arrival, the erasure probability is chosen from $\mathbf{P}$
- Erasure probability remains fixed until the next success

Alternative Problem: $P(i)$ is picked with probability $Q(i)$

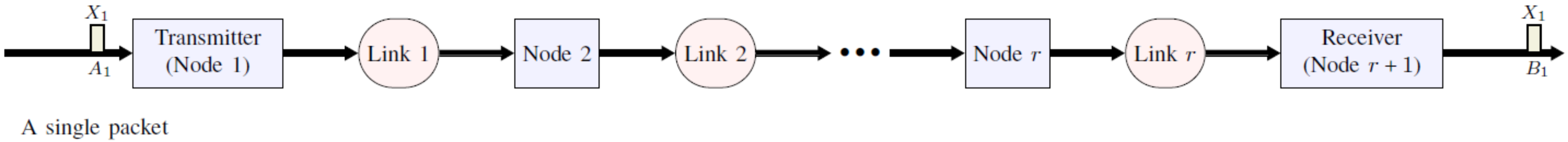
- The capacity of the alternative setting is higher (similar to waiting-time paradox)

1 Packet over r Link ⇔ r Packets over 1 Link

Classical:



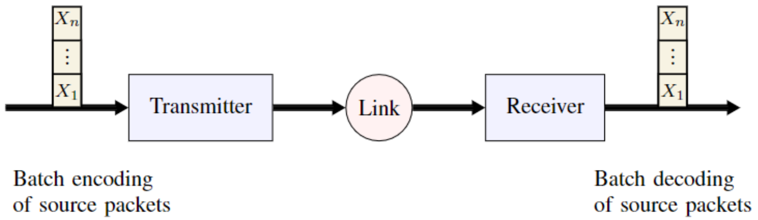
Dual:



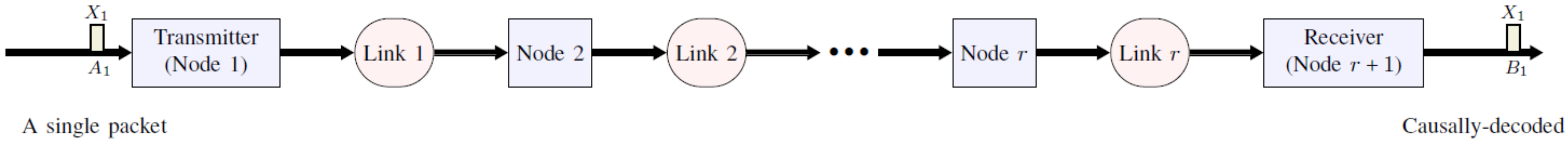
What about other (non-erasure) links?

1 Packet over r Link $\Leftrightarrow$ r Packets over 1 Link

Classical:



Dual:



What about other (non-erasure) links?

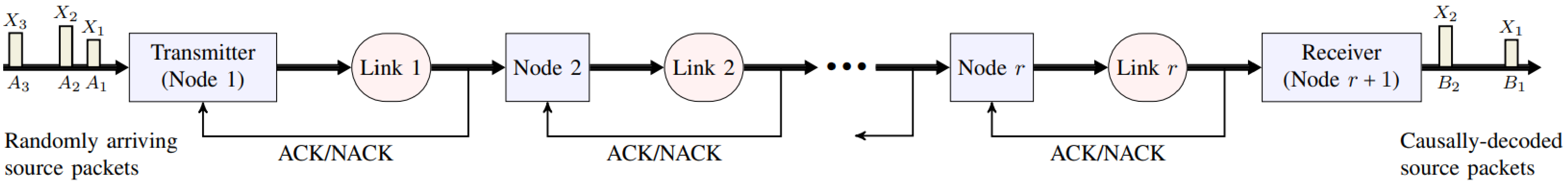
[Ling–Scarlett IT'22]: For a single bit transmitted over a cascade of homogeneous BSCs,

$$\vec{V} > C$$

for $p \to 0$.

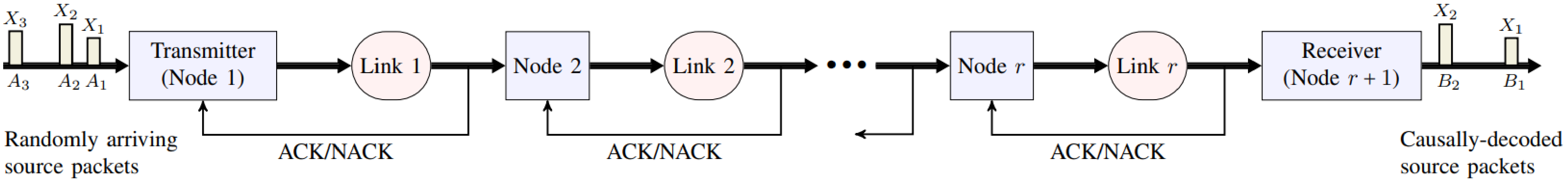
Stream of Causally Arriving Packets

System Model: Reminder



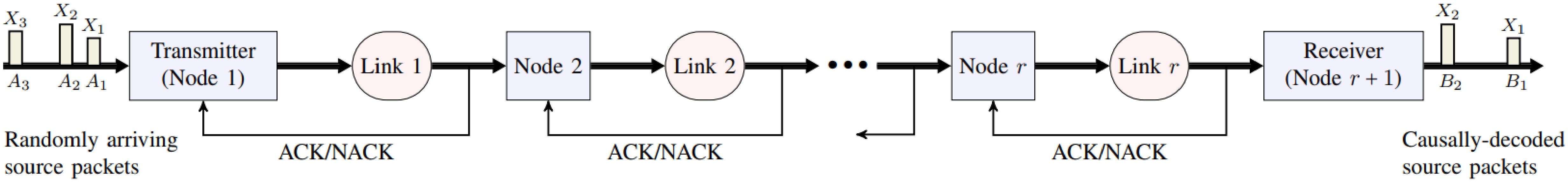
- Cascade of **packet-erasure** links
- IID **Bernoulli erasures** in each link
- Erasures are independent across different links (and times)
- Instantaneous perfect acknowledgment (ACK) feedback
- **Stream of causally arriving packets** (at random, periodically, …)
- In-order communications

System Model: Reminder



- Source stream: Packet $m \in \mathbb{Z}$ arrives at time $A_m \in \mathbb{Z}$
- Output of link i serves as the input to node $i+1$
- At each time step: Packet sent in link i is erased with probability p_i
  - Packets are acknowledged upon arrival
  - If packet erased, it is retransmitted until successful arrival at next node
- Departure process
  - Packet $m \in \mathbb{Z}$ arrives at final receiver at time $B_m \in \mathbb{Z}$

System Model: Reminder

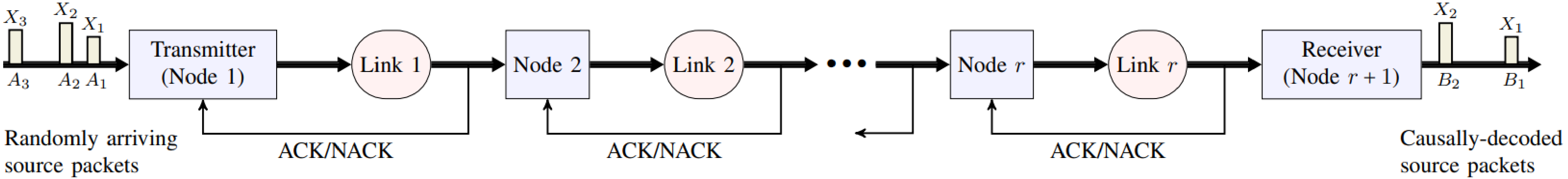


- **Arrive-failure probability:** $P_\mathrm{e}(N) \triangleq \sup_{m\in\mathbb{Z}} \Pr(B_m > A_m + N)$

- **Information velocity:** $\vec{V} \triangleq \sup\left\{\alpha > 0 \middle| r = \lceil \alpha N \rceil, \lim_{N\to\infty} P_e(N) = 0\right\}$
  - $r \propto N$ is the proper growth rate

- **Error exponent:** $E \triangleq \lim_{N\to\infty} -\frac{1}{N} \log P_e(N)$ for $\alpha < \vec{V}$

System Model: Reminder

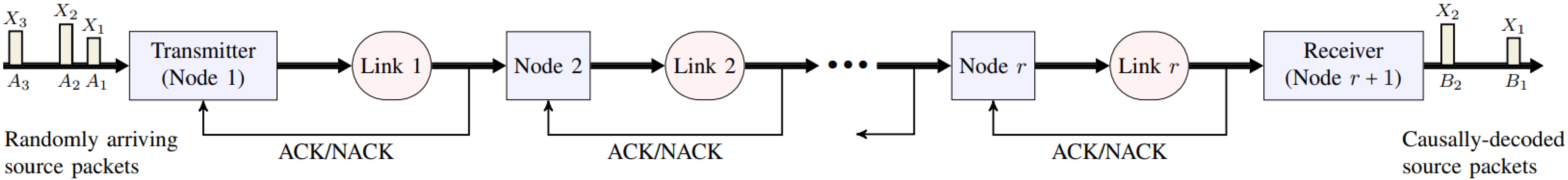


- We know how the first packet behaves
- The second packet will arrive slower (probabilistically)
- The third even slower

⋮

- Steady state—we will concentrate on this

System Model: Queuing Theory Perspective



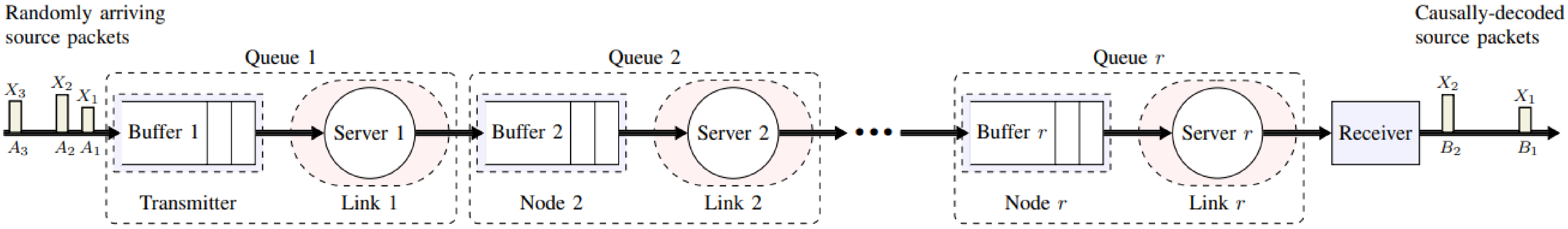
- **Input stream:** positive arrival (average) rate $\lambda > 0$
- Condition for **stable steady-state system**: $\lambda < 1 - p_i \quad \forall i \in \{1, \dots, r\}$
- We will concentrate on homogeneous case: $p_1 = p_2 = \cdots = p_r \triangleq p$

Arrival processes:

- IID Bernoulli arrivals—by lifting the results for a single packet
- Stationary ergodic process
- Periodic (deterministic) arrivals

} by converging to IID Bernoulli arrivals…

System Model: Queuing Theory Perspective



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Arrival processes:

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} by converging to IID Bernoulli arrivals…

Burke's Theorem

- IID $\mathcal{Ber}(1-p)$ service times $\Leftrightarrow$ IID $\mathcal{Geo}(1-p)$ interservice times
- Waiting time = queueing time + service time

Theorem [Hsu–Burke TCOM'76][Pujolle–Claude–Seret '86][Desert–Daduna '02]:

Assume a **single** queue in **steady state** with

- IID $\mathcal{Ber}(\lambda)$ arrival times $(\lambda < 1-p)$
- IID $\mathcal{Ber}(1-p)$ service times

$\Rightarrow$ IID $\mathcal{Ber}(\lambda)$ departure times

$\Rightarrow$ # packets in queue at time t is independent of departure process prior to time t

$\Rightarrow$ Waiting time of a packet $\sim \mathcal{Geo}\left(1-\frac{p}{1-\lambda}\right) \perp\!\!\!\perp$ departure process before packet's departure

- Discrete-time analogue of classical results of [Burke '56][Reich '57][Jackson '54]

Reich's Theorem for Tandem Queues

Theorem [Hui '90][Prabhakar–Gallager TIT'03]:

Cascade of r queues in **steady state** with

- IID $\mathcal{Ber}(\lambda)$ arrival times $(\lambda < 1 - p \quad \forall i \in \{1, \dots r\})$
- IID $\mathcal{Ber}(1 - p)$ service times at server i
- Service times are independent across servers (and within)

⇒ Waiting times of a packet are independent across queues

⇒ Waiting time at queue $i \sim \mathcal{Geo}\left(1 - \frac{p}{1-\lambda}\right)$

⇒ # packets in different queues are independent at a given time

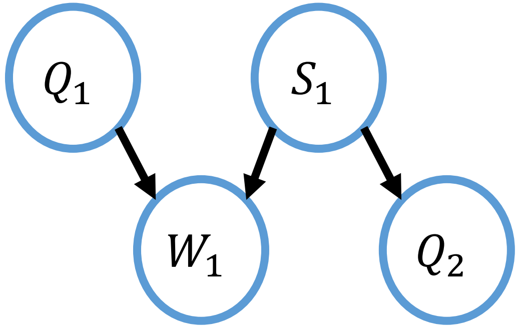
- Discrete-time analogue of a classical result of [Reich '57]

Surprising Hidden Implication [Burke ’63]

Waiting time = queueing time + service time

$$W_1 = Q_1 + S_1$$
$$W_2 = Q_2 + S_2$$

- The waiting times are independent: $W_1 \perp\!\!\!\perp W_2$
- The service times are independent: $S_1 \perp\!\!\!\perp S_2$
- Service times are independent of respective queueing times: $S_1 \perp\!\!\!\perp Q_1, S_2 \perp\!\!\!\perp Q_2$
- **But the queueing times are dependent!** $Q_1 \not\perp\!\!\!\perp Q_2$
- **Proof:** By direct calculation, $\Pr(Q_2 = 0 | Q_1 = 0) > \Pr(Q_2 = 0)$. Alternatively,
  - $S_2 \perp\!\!\!\perp (Q_1, Q_2, S_1)$ and $Q_2 + S_2 = W_2 \perp\!\!\!\perp W_1 \Rightarrow Q_2 \perp\!\!\!\perp W_1$
  - $Q_2 \not\perp\!\!\!\perp S_1$ since for $S_1 \to \infty,\ Q_2 \to 0$
  - $Q_2 \perp\!\!\!\perp W_1 = S_1 + Q_1$



Cascade of r Independent Links, IID Bernoulli Arrivals

Corollary [Domanovitz–Philosof–K. INFOCOM'22]:

Cascade of r queues in **steady state** with

- IID $\mathcal{Ber}(\lambda)$ arrival times $(\lambda < 1 - p)$
- IID $\mathcal{Ber}(1 - p)$ service times at server i
- Service times are independent across servers (and within)

⇒ **Homogenous links:** Replace p with $\frac{p}{1-\lambda}$ in single-packet expressions:

- **IV:** $\vec{V} = 1 - \frac{p}{1-\lambda}$
- **EE:** for $\alpha < \vec{V}$: $E = \mathbb{D}\left(\alpha \| 1 - \frac{p}{1-\lambda}\right)$

Heterogeneous links: Replace p_i with $\frac{p_i}{1-\lambda}$ in single-packet expressions

Stationary Ergodic Arrival Process

- Assume now a stationary ergodic arrival process of rate λ

Theorem [Mountford–Prabhakar '95]:

- Homogeneous queues

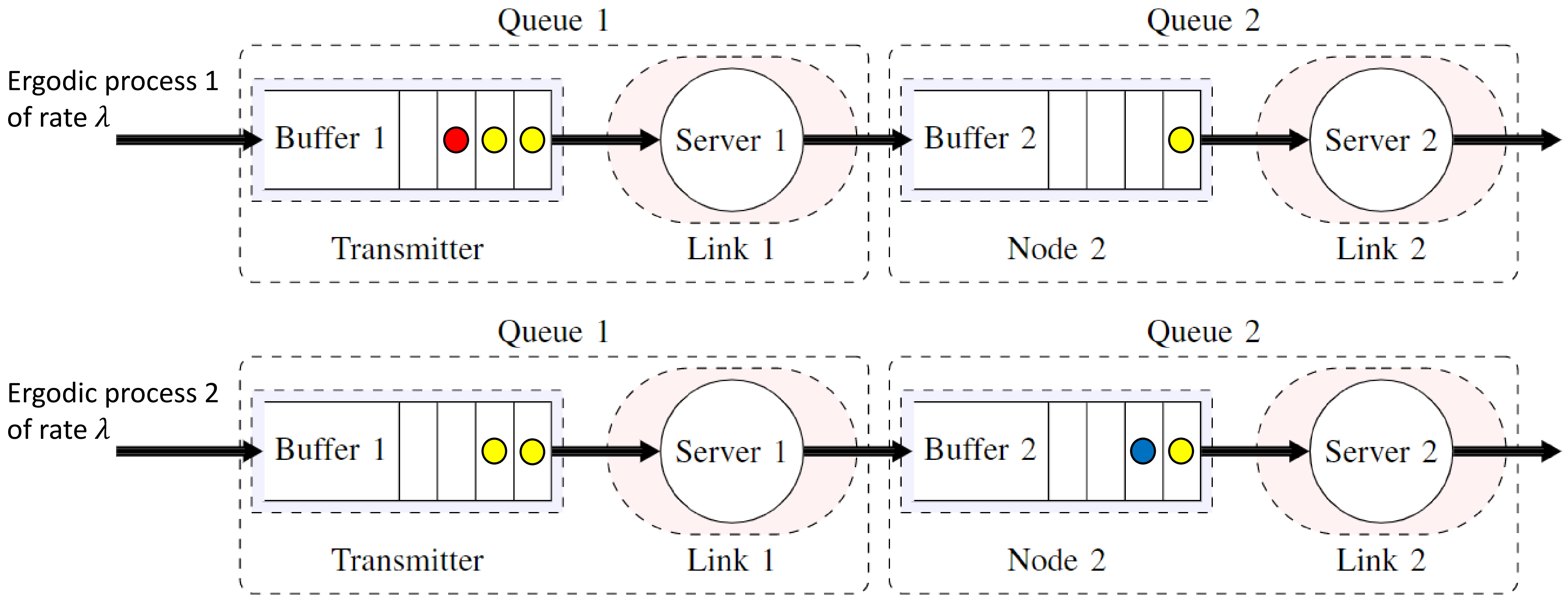
For $r \to \infty$, the departure process converges to IID $\mathcal{Ber}(\lambda)$ arrivals

- Proof uses the coupling technique of [Anantharam '93]

Stationary Ergodic Arrival Process: Illustration

Event: Initial state of the system

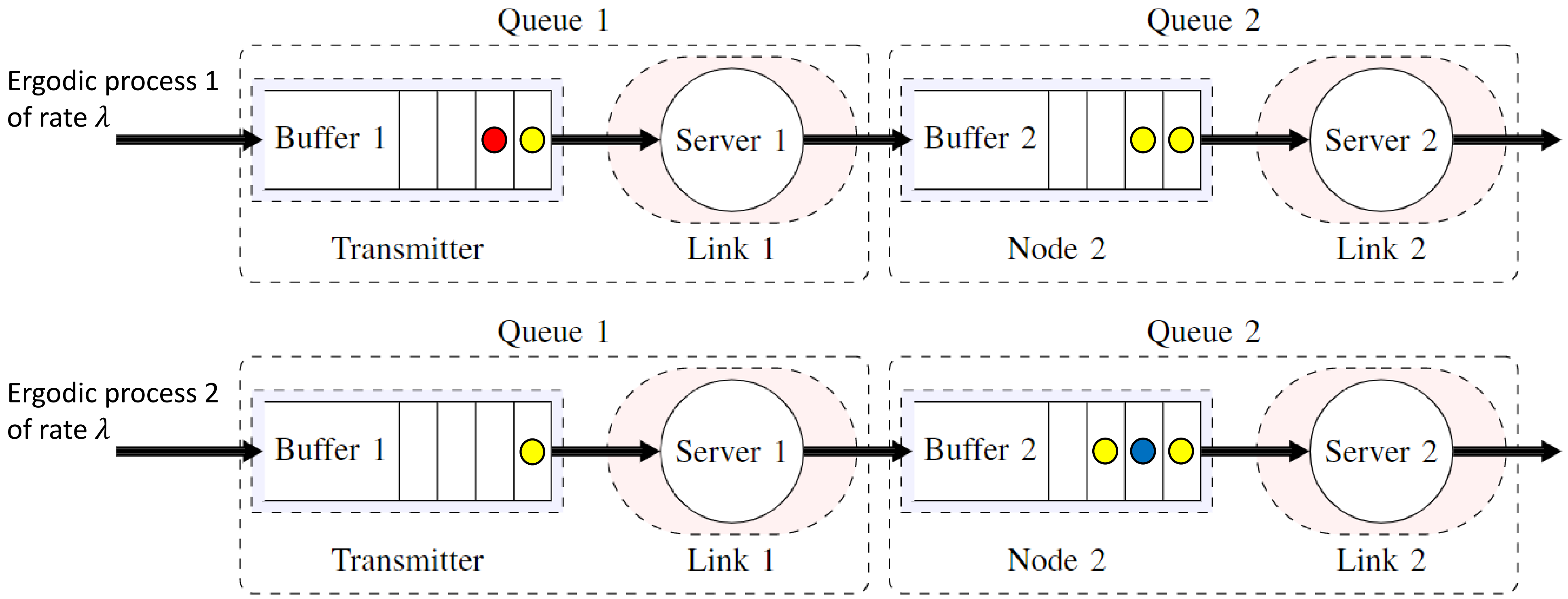
Next: Service @ Server 1



Stationary Ergodic Arrival Process: Illustration

Event: Service @ Server 1

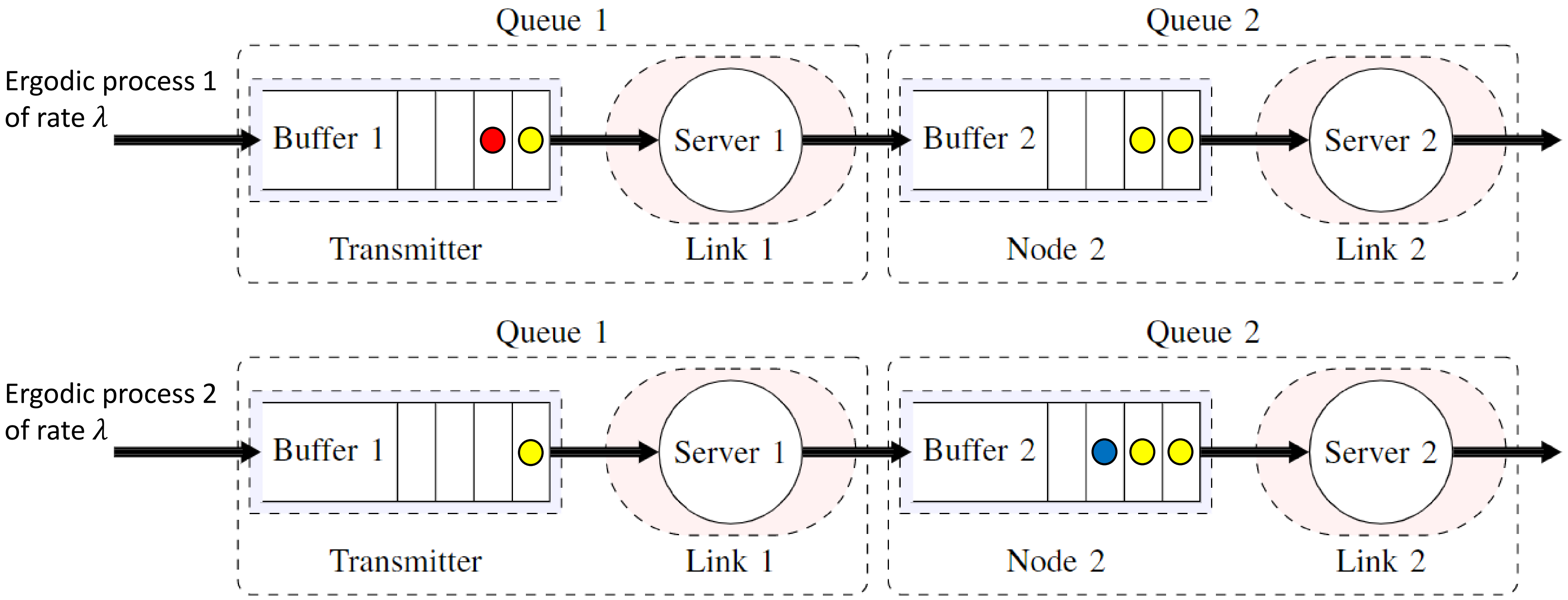
Next: Recoloring @ Server 2



Stationary Ergodic Arrival Process: Illustration

Event: Recoloring @ Server 2

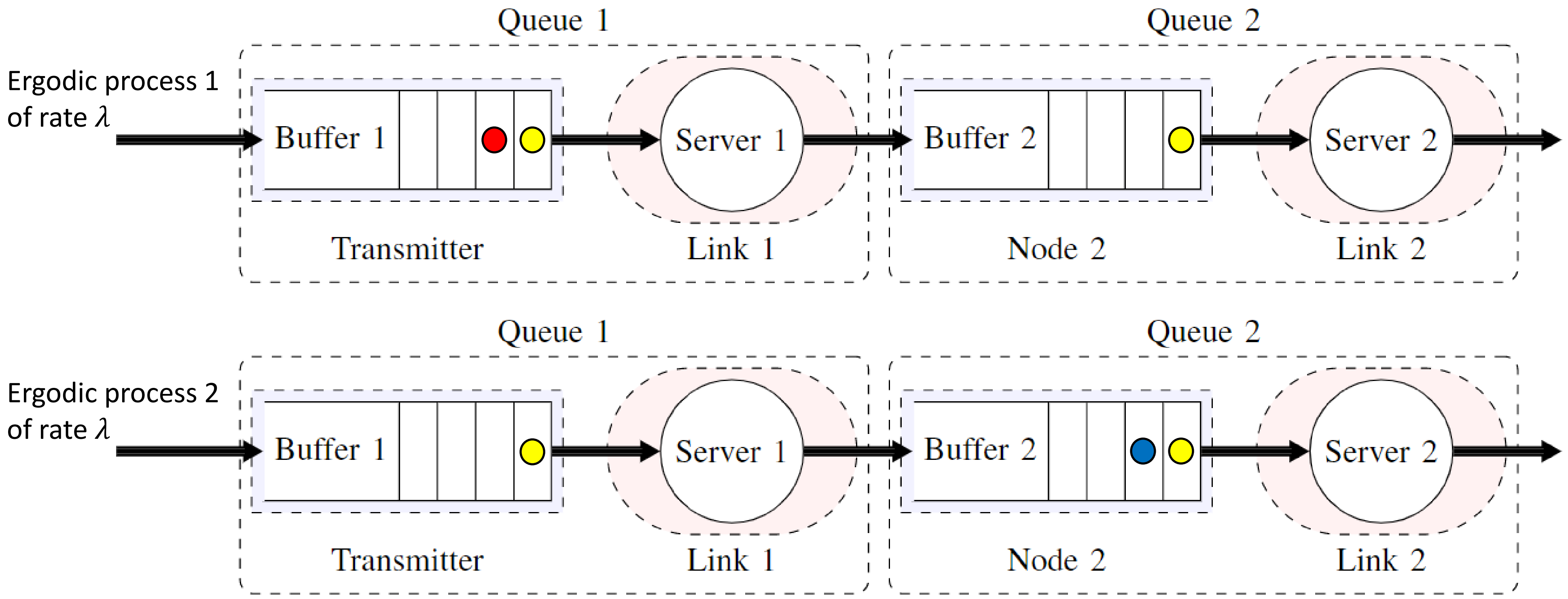
Next: Service @ Server 2



Stationary Ergodic Arrival Process: Illustration

Event: Service @ Server 2

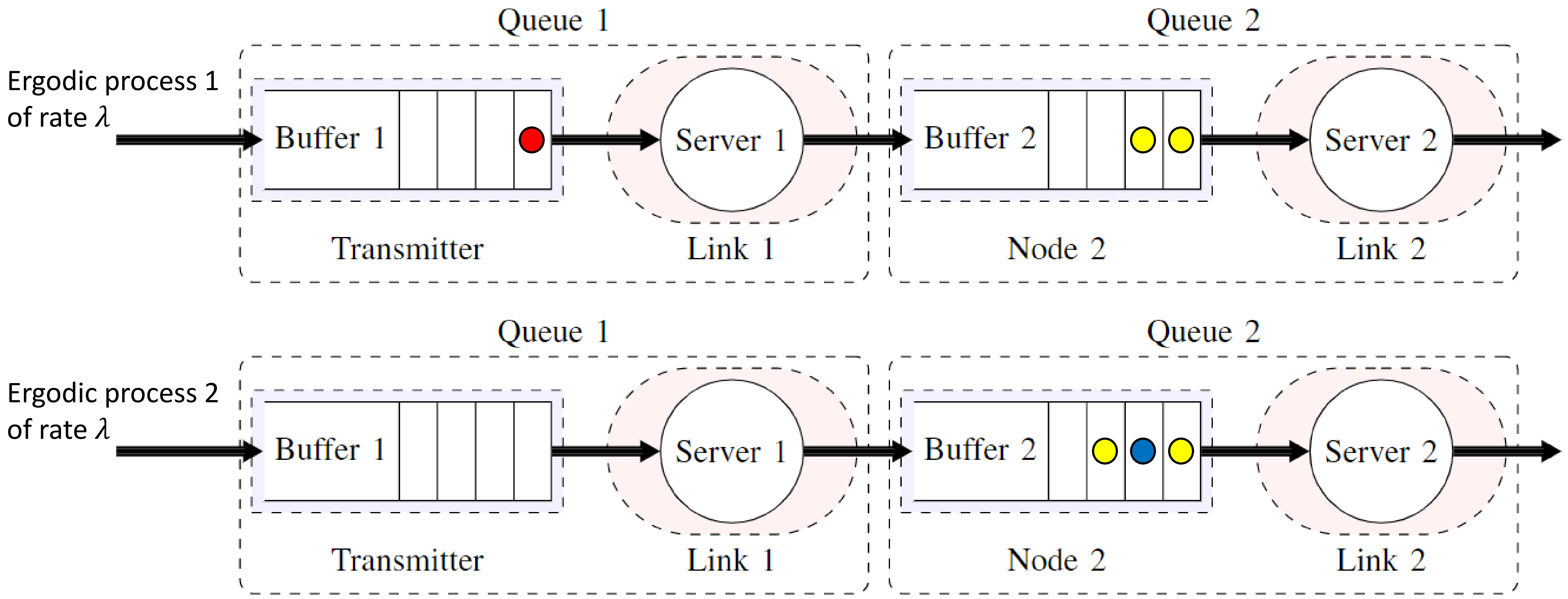
Next: Service @ Server 1



Stationary Ergodic Arrival Process: Illustration

Event: Service @ Server 1

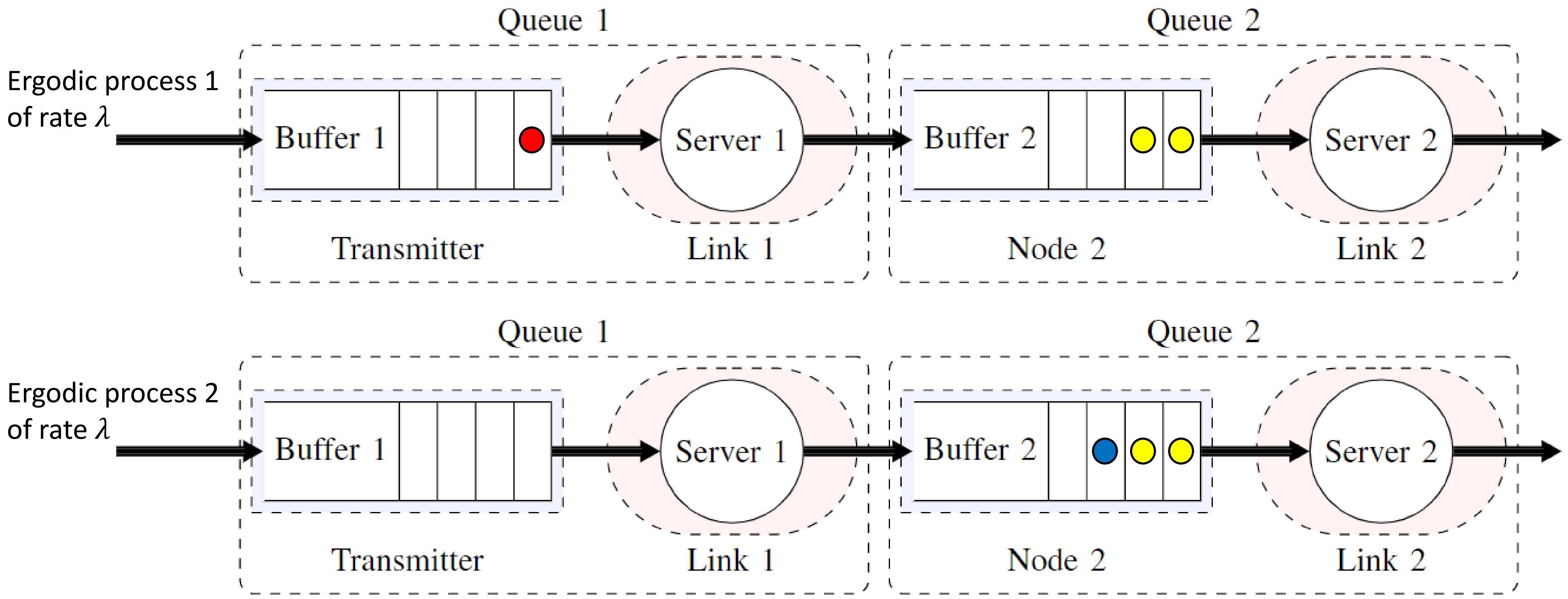
Next: Recoloring @ Server 2



Stationary Ergodic Arrival Process: Illustration

Event: Recoloring @ Server 2

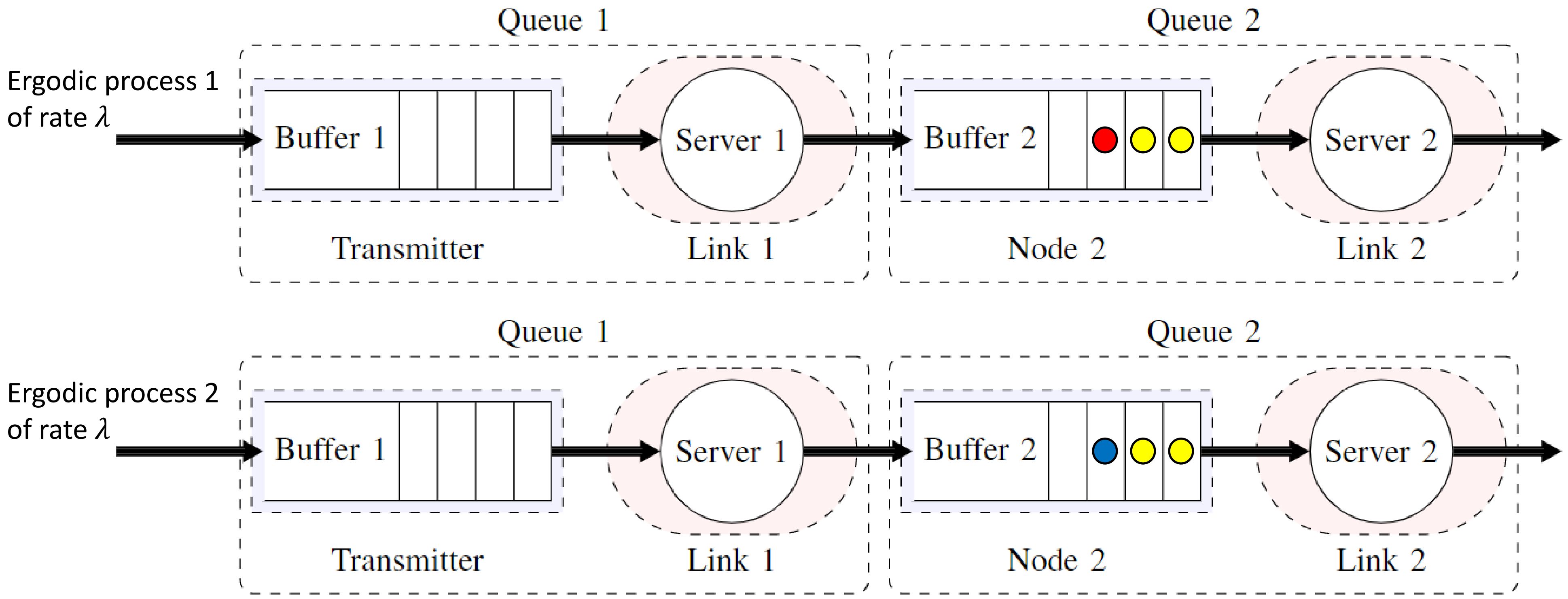
Next: Service @ Server 1



Stationary Ergodic Arrival Process: Illustration

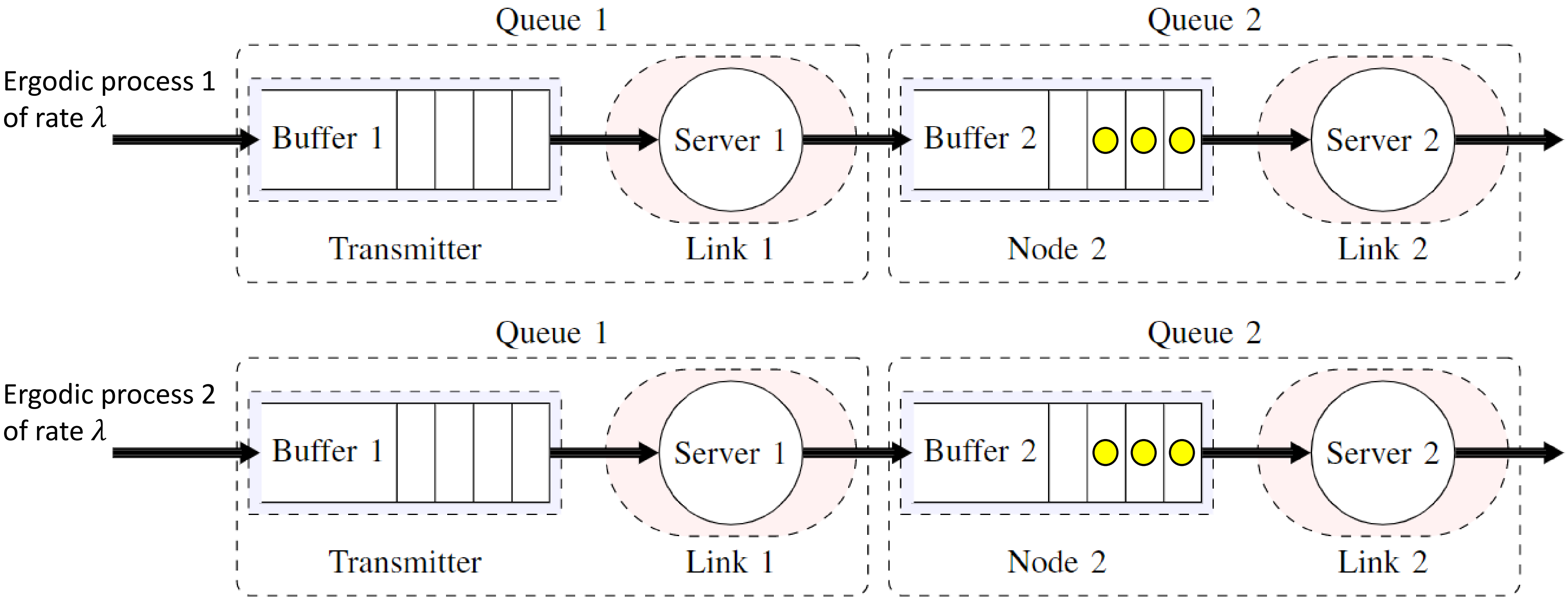
Event: Service @ Server 1

Next: Merge: Red + Blue = Yellow



Stationary Ergodic Arrival Process: Illustration

Event: Merge: Red + Blue = Yellow



Stationary Ergodic Arrival Process: Proof Idea

For any two ergodic input processes of the same rate λ

- Fraction of yellows increases with each queue
- Fraction of yellows $\rightarrow 1$ for $r \rightarrow \infty$

  $\Rightarrow$ Output processes converge to the same process

- Take one the input processes to have IID $\mathcal{Ber}(\lambda)$ arrivals

  $\Rightarrow$ By Burke's theorem the output of Server 1 has also IID $\mathcal{Ber}(\lambda)$ arrivals

  $\Rightarrow$ Holds also for all subsequent servers

$\Rightarrow$ Any ergodic input process converges to IID $\mathcal{Ber}(\lambda)$ arrivals

Stationary Ergodic Arrival Process

- Is that enough to derive IV and EE results?

Stationary Ergodic Arrival Process

- Is it enough to derive IV and EE results? No ☹
- It suffices to prove impossibility (converse) results
- For achievability: Requires bounding the effect of red/blue packets

  or a robustness/continuity result

Cascade of r Independent Links, Ergodic Arrival Process

Theorem [Domanovitz–Philosof–K.]:

Cascade of r queues in **steady state** with

- Ergodic arrival times of rate $\lambda < 1 - p$
- IID $\mathcal{Ber}(1-p)$ service times at server i
- Service times are independent across servers (and within)

- **Homogenous links:** Replace p with $\frac{p}{1-\lambda}$ in single-packet expressions
  - **IV:** $\vec{V} \overset{=}{\leq} 1 - \frac{p}{1-\lambda}$ (same as for IID Bernoulli arrivals)
  - **EE:** for $\alpha < \vec{V}$: $E \leq \mathbb{D}\left(\alpha \| 1 - \frac{p}{1-\lambda}\right)$ (upper bounded by EE of IID Bernoulli arrivals)

- Proof based on the technique of [Mountford–Prabhakar '95]

Heterogeneous links: Similar results apply.

Periodic (Deterministic) Arrivals

- Assume now periodic arrivals: A packet arrives every $1/\lambda$ time steps

Extension of last theorem [Domanovitz–Philosof–K.]:

- Homogeneous/heterogeneous queues

$\Rightarrow$ For $r \to \infty$, the departure process converges to IID $\mathcal{Ber}(\lambda)$ arrivals

Remark: Can be extended to cyclostationary cycloergodic processes

- Cycloergodic processes [Boyles–Gardner IT'83].
- More generally, extends to any input process whose long-time average converges to a deterministic value a.s.

Cascade of r Independent Links, Periodic Arrivals

Corollary [Domanovitz–Philosof–K.]:

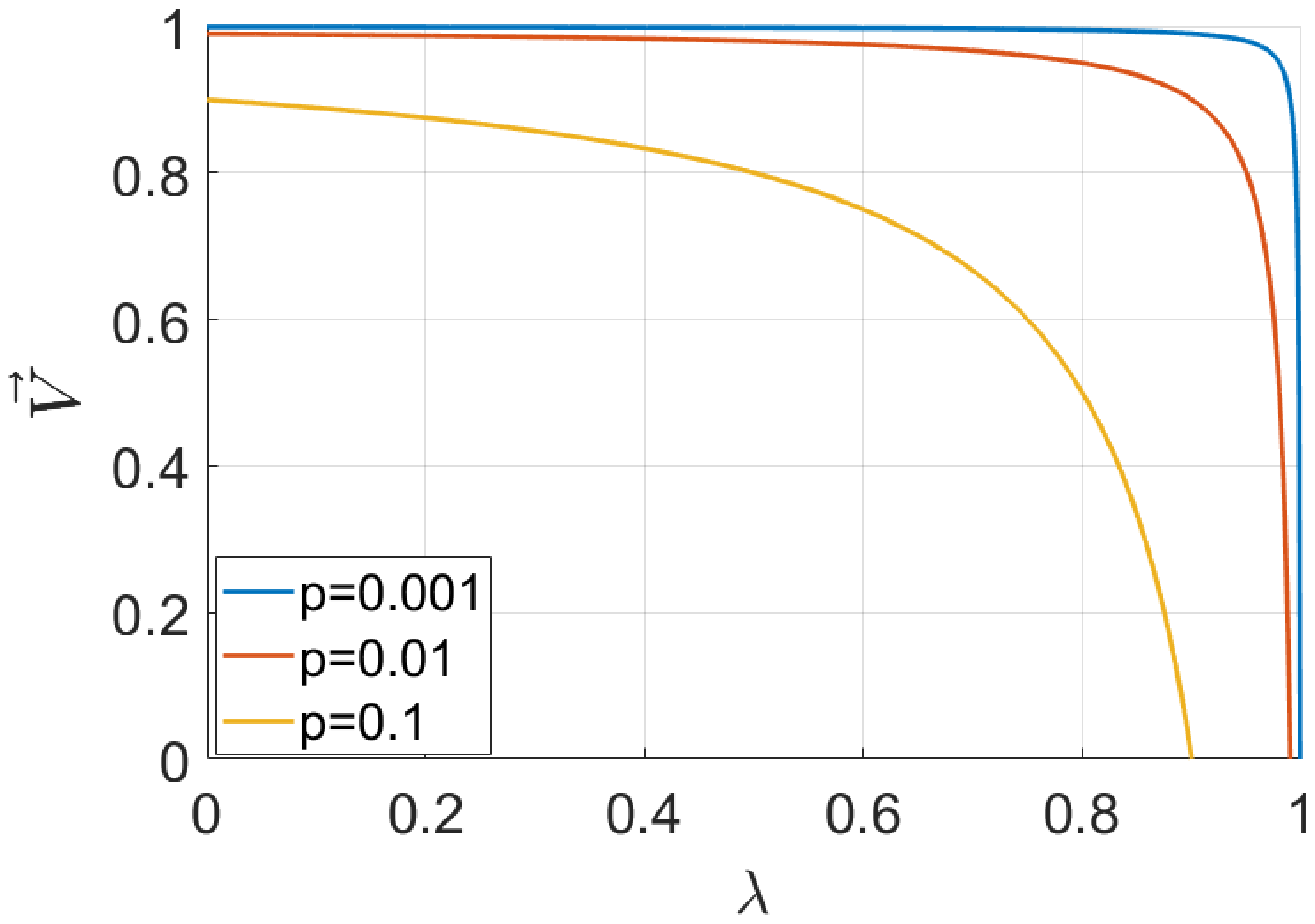
- **Homogenous links:** Replace p with $\frac{p}{1-\lambda}$ in single-packet expressions
  - **IV:** $\vec{V} \overset{=}{\leq} 1 - \frac{p}{1-\lambda}$ (same as for IID geometric interarrivals)
  - **EE:** for $\alpha < \vec{V}$: $E \overset{=}{\leq} \mathbb{D}\left(\alpha \| 1 - \frac{p}{1-\lambda}\right)$ (same as for IID geometric interarrivals)

- **Heterogeneous links:** Replace p_i with $\frac{p_i}{1-\lambda}$ in single-packet expressions

  (same as for IID geometric interarrivals)

Numerical Results: IV versus Arrival Rate

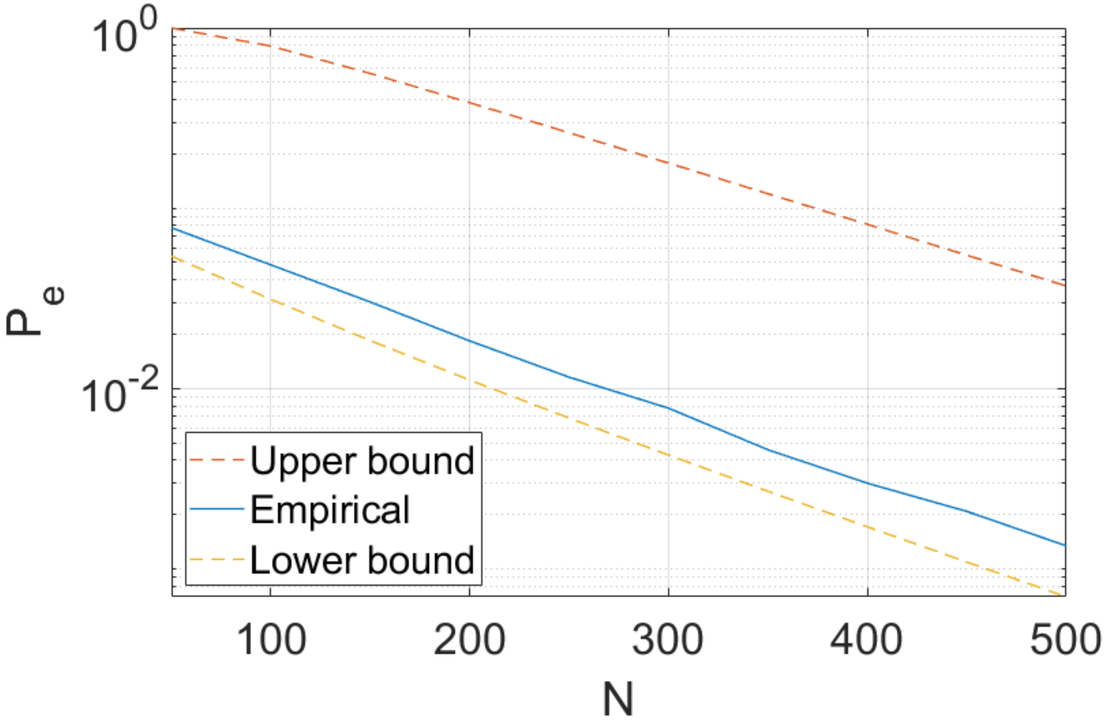
- Homogeneous links with $p = 0.001, 0.01, 0.1$



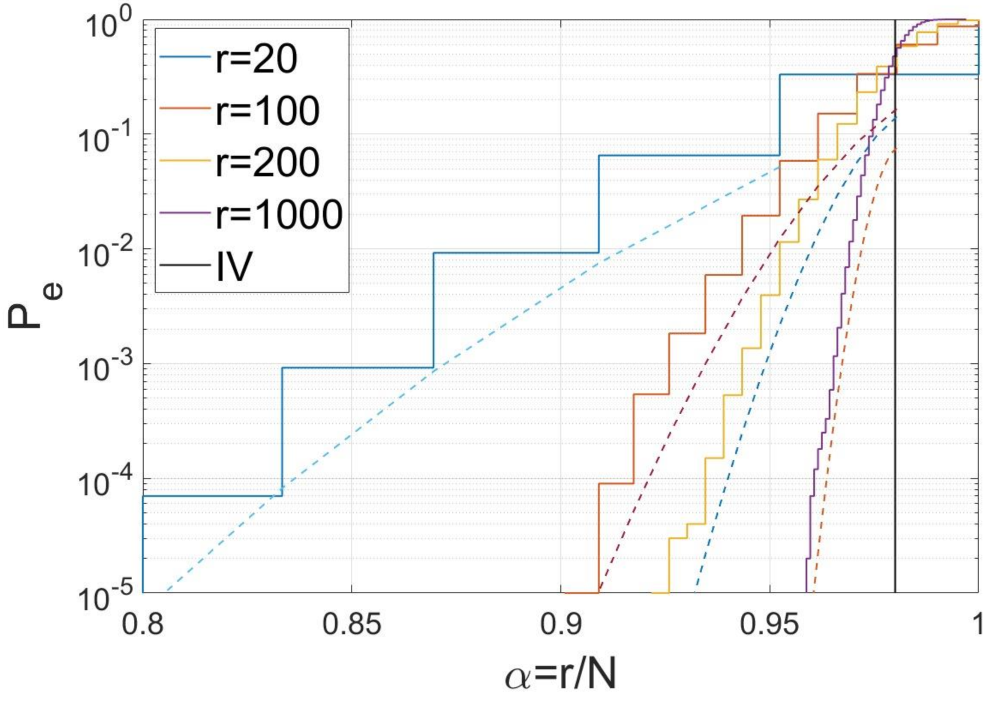
Numerical results: Arrive-Failure Probability

- Homogeneous links with $p = 0.01$
- $\lambda = 0.5$
- $\vec{V} = 1 - p = 0.99$

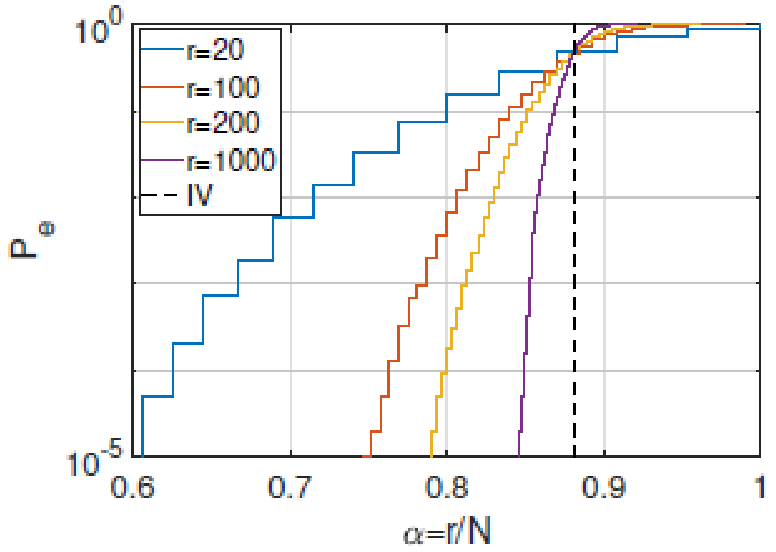
- $\alpha = 0.96$



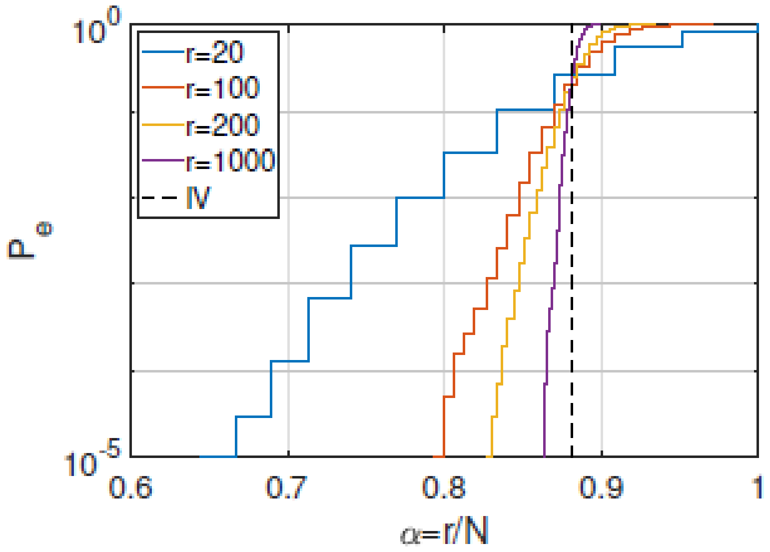
- $r = 20, 100, 200, 1000$ relays



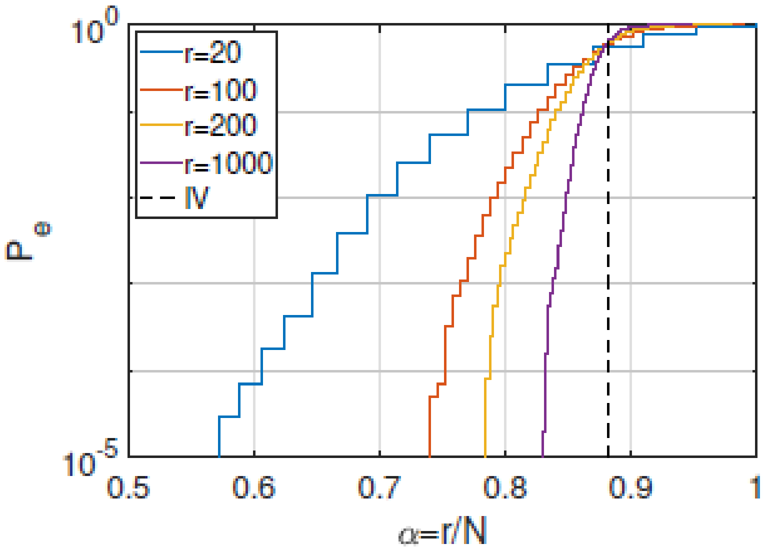
Numerical Results: Different Arrival Processes



(a) Geometric i.i.d. interarrival times.



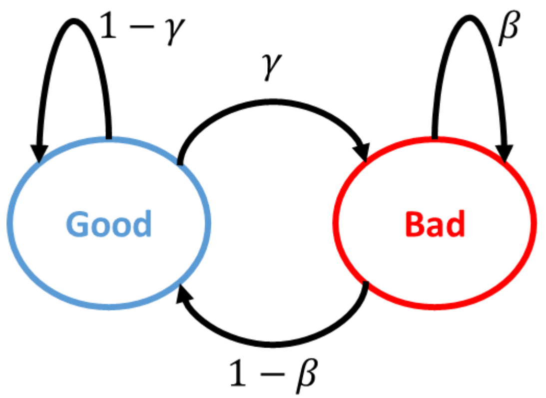
(b) Deterministic interarrival times.



(c) Gilbert–Eliott arrival process with $\gamma = 0.01$, $\beta = 0.1$, $\varepsilon = 0.45$.

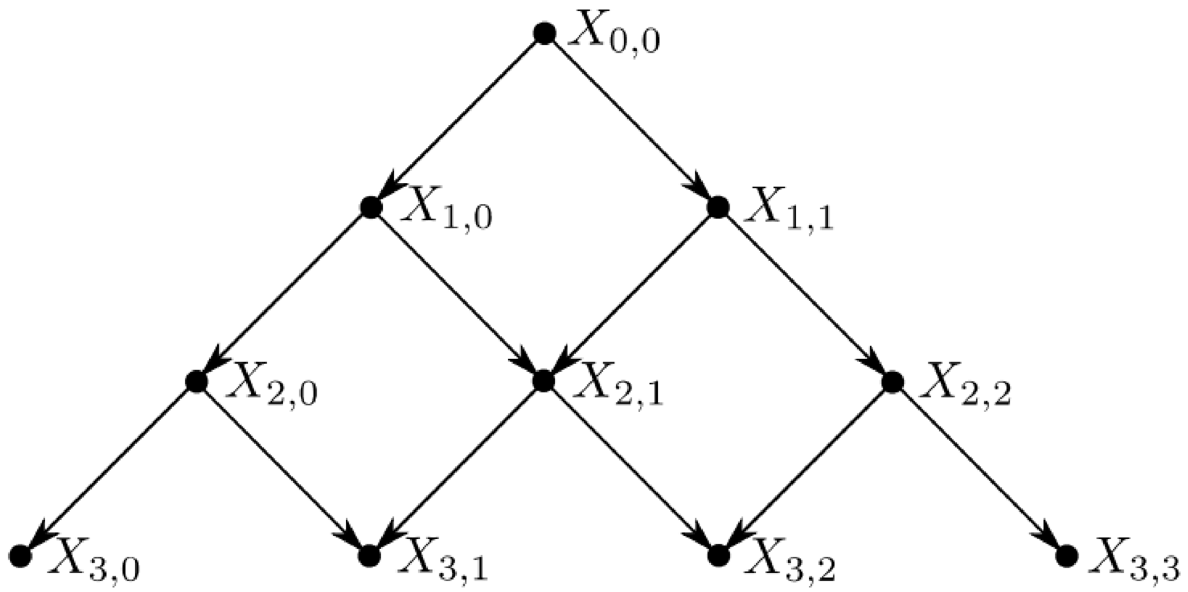
- $\lambda = 0.5, \mathbf{P} = (0.01, 0.1), \mathbf{Q} = (0.5, 0.5)$

- **Gilbert–Elliott model:** Two-state Markov model
  - @Good state—packet arrives with probability ϵ
  - @Bad state—packet arrives with probability 1



Extensions

- Heterogeneous links (@backup slides & paper)
- Several servers at each queue
- 2D regular grids



[Makur–Mossel–Polyanskiy TIT ‘22]

Discussion & Future Research

- We assumed feedback ⇒ What is the IV without/with delayed feedback?
  - We know the answer for a single packet since feedback is not needed
- We studied the behavior of a single/first packet and in stead state
- What is the IV for intermediate (“transient”) packets?
- All the results are easily adaptable to the continuous-time setting
  - In fact the queueing theory results were originally derived for this setting
  - “Poisson IV”
- We assumed *in-order* transmission of *all* packets
  - “Slow” packets can be thrown away to improve overall performance
- “Anytime *Anywhere* Reliability”—important if data pertain to control
- Other channels with/without feedback & more general networks

IV League

Backup Slides

Single Packet, Homogeneous Links

Bounds on P_e via binomial coefficient bounds:

$$P_e(N) = \Pr\left(\sum_{i=1}^{r} \tau_i > N\right) = \sum_{j=N+1}^{\infty} \binom{j-1}{r-1} (1-p)^r p^{r-j}$$

Using the entropy bounds on binomial coeff. [Ash's book]: $H_b(q) \triangleq -q \log q - (q-1) \log(1-q)$

$$\frac{1}{2} \exp\left\{nH_b\left(\frac{k}{n}\right)\right\} \sqrt{\frac{n}{2k(n-k)}} \le \binom{n}{k} \le \frac{1}{\sqrt{\pi}} \exp\left\{nH_b\left(\frac{k}{n}\right)\right\} \sqrt{\frac{n}{2k(n-k)}},$$

yields

$$P_e(N) \le \sum_{\ell=N}^{\infty} \frac{(1-p)\sqrt{\ell} \exp\left\{-\ell \cdot \mathbb{D}\left(\frac{r-1}{\ell} \middle\| 1-p\right)\right\}}{\sqrt{2\pi(r-1)(\ell+1-r)}} \le \sum_{\ell=N}^{\infty} \frac{(1-p)\sqrt{N} \exp\left\{-\ell \cdot \mathbb{D}\left(\frac{r-1}{N} \middle\| 1-p\right)\right\}}{\sqrt{2\pi(r-1)(N+1-r)}}$$

$$= \frac{(1-p)\sqrt{N}}{\sqrt{2\pi(r-1)(N-r+1)}} \cdot \frac{\exp\left\{-N \cdot \mathbb{D}\left(\frac{r-1}{N} \| 1-p\right)\right\}}{1-\exp\left\{-N \cdot \mathbb{D}\left(\frac{r-1}{N} \| 1-p\right)\right\}}$$

Single Packet, Homogeneous Links

Bounds on P_e via binomial coefficient bounds:

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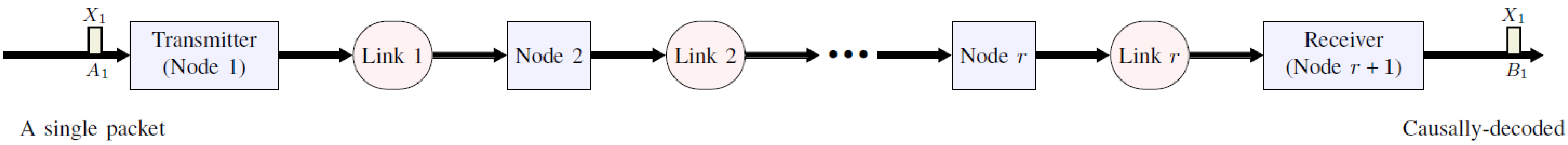
yields

$$P_e(N) \geq \sum_{\ell=N}^{\infty} \frac{(1-p)\sqrt{\ell}\exp\left\{-\ell\cdot\mathbb{D}\left(\frac{r-1}{N}\middle\|1-p\right)\right\}}{\sqrt{2\times 4(r-1)(\ell+1-r)}} \geq \frac{(1-p)\sqrt{N}\exp\left\{-N\cdot\mathbb{D}\left(\frac{r-1}{N}\middle\|1-p\right)\right\}}{\sqrt{8(r-1)(N+1-r)}}$$

Single Packet, Heterogeneous Links



Heterogeneous links: the erasure probabilities of different links—$p(1), p(2), \dots p(r)$—may differ

- We will consider two settings for the erasure probabilities:

1. **Fixed channels-type:** Fraction ("type") $Q(i)$ of all r channels have erasure probability $P(i)$
   - Assume only $S < \infty$ possible channel erasure probabilities (can be lifted):

$$\mathbf{Q} \triangleq [Q(1) \quad \cdots \quad Q(S)], \qquad \mathbf{P} \triangleq [P(1) \quad \cdots \quad P(S)]$$

Remark: Order of $p(1), p(2), \dots p(r)$ given a certain type $\mathbf{Q}$ doesn't matter

2. **Probabilistic setting:** $p(1), p(2), \dots p(r) \in \mathbf{P} \sim \text{IID}$ according to $\tilde{\mathbf{Q}}$

Single Packet, Heterogeneous Links: Fixed Channels-Type

1. **Fixed channels-type:**

   Fraction ("type") $Q(i)$ of all r channels have erasure probability $P(i)$

$$\mathbf{Q} \triangleq [Q(1) \quad \cdots \quad Q(S)], \qquad \mathbf{P} \triangleq [P(1) \quad \cdots \quad P(S)]$$

Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **IV:** $\vec{V} = 1/\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)}$

- **EE, 1st expression:** for $\alpha < \vec{V}$: $E^{\text{fixed}}(\mathbf{Q}) = (1-\alpha)\log x + \alpha \sum_{i=1}^{S} Q(i) \log \frac{1-P(i)x}{1-P(i)}$

  where $x \in (1, 1/\max \mathbf{P})$ is the solution of $\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)x} = \frac{1}{\alpha}$

Corollary: For $r = o(N)$, EE equals that of worst link: $E = -\log \max \mathbf{P}$.

Single Packet, Heterogeneous Links: Fixed Channels-Type

Proof of IV characterization: Define

- $\alpha \triangleq r/N$
- $R(i) \triangleq$ # links with erasure probability $P(i) \;\Leftrightarrow\; Q(i) \equiv R(i)/r$

$$\lim_{N\to\infty} P_e(N) = \lim_{N\to\infty} \Pr\left(\frac{1}{r}\sum_{\ell=1}^{r} \tau_\ell > \frac{N}{r}\right) = \lim_{N\to\infty} \Pr\left(\sum_{i=1}^{S} \frac{R(i)}{r} \cdot \frac{1}{R(i)} \sum_{\ell : p_\ell = P(i)} \tau_\ell > \frac{N}{r}\right)$$

$$= \begin{cases} 0, & \displaystyle\sum_{i=1}^{S} Q(i) \frac{1}{1-P(i)} < 1/\alpha \\ 1, & \displaystyle\sum_{i=1}^{S} Q(i) \frac{1}{1-P(i)} > 1/\alpha \end{cases} \quad \Rightarrow \quad \vec{V} = \frac{1}{\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)}}$$

Single Packet, Heterogeneous Links: Fixed Channels-Type

Proof of (1st) EE expression: By the Gärtner–Ellis theorem

$$E^{\text{fixed}} = -\sup_{\nu>0}\left\{\nu - \alpha\sum_{i=1}^{S} Q(i)\log M_i(\nu)\right\} = (1-\alpha)\log x + \alpha\sum_{i=1}^{S} Q(i)\log\frac{1-P(i)x}{1-P(i)}$$

where

- $M_i(\nu) = \frac{\left(1-P(i)\right)\exp\{\nu\}}{1-P(i)\exp\{\nu\}}$ is the moment-generating function of $\mathcal{Geo}\left(1-P(i)\right)$
- $x \in (1, 1/\max \mathbf{P})$ is the solution of $\sum_{i=1}^{S}\frac{Q(i)}{1-P(i)x} = \frac{1}{\alpha}$

Chernoff's upper bound on P_e: Results in the above EE (achievable only)

Single Packet, Heterogeneous Links: Fixed Channels-Type

Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **IV:** $\vec{V} = 1/\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)}$
- **EE, 1st expression:** for $\alpha < \vec{V}$: $E^{\text{fixed}}(\mathbf{Q}) = (1-\alpha)\log x + \alpha \sum_{i=1}^{S} Q(i) \log \frac{1-P(i)x}{1-P(i)}$

  where $x \in (1, 1/\max \mathbf{P})$ is the solution of $\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)x} = \frac{1}{\alpha}$

- **EE, 2nd expression:** for $\alpha < \vec{V}$: $E^{\text{fixed}}(\mathbf{Q}) = \min_{\substack{U \in \Delta_{S-1} \\ U(i) \geq \frac{\alpha Q(i)}{1-P(i)} \ \forall i \in [S]}} \sum_{i=1}^{S} U(i) \mathbb{D}\left(\frac{\alpha Q(i)}{U(i)} \middle\| 1-P(i)\right)$

  where $\Delta_{S-1} \triangleq \left\{(x_1, \dots, x_S) \middle| \sum_{i=1}^{S} x_i = 1,\ x_i \geq 0 \ \ \forall i\right\}$ is the standard simplex

Remark: Both EE expressions can be proved to be equal also directly via the KKT conditions.

Single Packet, Heterogeneous Links: Fixed Channels-Type

Proof of 2nd EE expression:

- <u>Idea:</u> For each $P(i)$ and $Q(i)$, look at corresponding fraction $U(i)$ of delay steps
- *Upper bound:* Run over all possible $\mathbf{U}$

$$P_e(N) = \Pr\left(\sum_{i=1}^{r} \tau_i > N\right) \leq \sum_{U \in \mathcal{Q}_N} \Pr\left(\sum_{\ell : p_\ell = P(i)} \tau_\ell \geq NU(i) \;\; \forall i \in [S]\right)$$

$$\leq \sum_{U \in \mathcal{Q}_N} \prod_{i=1}^{S} \Pr\left(\sum_{\ell : p_\ell = P(i)} \tau_\ell \geq NU(i)\right)$$

$$\dot{\leq} (N+1)^S \exp\left\{- \min_{\substack{U \in \Delta_{S-1} \\ U(i) \geq \frac{\alpha Q(i)}{1-P(i)} \;\forall i \in [S]}} \sum_{i=1}^{S} U(i) \mathbb{D}\left(\frac{\alpha Q(i)}{U(i)} \middle\| 1 - P(i)\right)\right\}$$

- $\mathcal{Q}_N \triangleq$ All partitions of N balls into S cells, divided by N (the "type class")

Single Packet, Heterogeneous Links: Fixed Channels-Type

Proof of 2nd EE expression:

- Idea: For each $P(i)$ and $Q(i)$, look at corresponding fraction $U(i)$ of delay steps
- *Lower bound:* Take "worst" possible $\mathbf{U}$

$$P_e(N) = \Pr\left(\sum_{i=1}^{r} \tau_i > N\right) \geq \max_{U \in \mathcal{Q}_N} \Pr\left(\sum_{\ell : p_\ell = P(i)} \tau_\ell > NU(i) \;\; \forall i \in [S]\right)$$

$$\geq \prod_{i=1}^{S} \Pr\left(\sum_{\ell : p_\ell = P(i)} \tau_\ell > NU(i)\right)$$

$$\dot{\geq} \exp\left\{- \min_{\substack{U \in \Delta_{S-1} \\ U(i) \geq \frac{\alpha Q(i)}{1-P(i)} \; \forall i \in [S]}} \sum_{i=1}^{S} U(i) \mathbb{D}\left(\frac{\alpha Q(i)}{U(i)} \middle\| 1 - P(i)\right)\right\}$$

- $\mathcal{Q}_N \triangleq$ All partitions of N balls into S cells, divided by N (the "type class")

Single Packet, Heterogeneous Links: Fixed Channels-Type

1. **Fixed channels-type:**

   Fraction ("type") $Q(i)$ of all r channels have erasure probability $P(i)$

$$\mathbf{Q} \triangleq [Q(1) \quad \cdots \quad Q(S)], \qquad \mathbf{P} \triangleq [P(1) \quad \cdots \quad P(S)]$$

Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **IV:** $\vec{V} = 1/\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)}$

- **EE, 1st expression:** for $\alpha < \vec{V}$: $E^{\text{fixed}}(\mathbf{Q}) = (1-\alpha)\log x + \alpha \sum_{i=1}^{S} Q(i) \log \frac{1-P(i)x}{1-P(i)}$

  where $x \in (1, 1/\max \mathbf{P})$ is the solution of $\sum_{i=1}^{S} \frac{Q(i)}{1-P(i)x} = \frac{1}{\alpha}$

Corollary: For $r = o(N)$, EE equals that of worst link: $E = -\log \max \mathbf{P}$.

Single Packet, Heterogeneous Links: Probabilistic Setting

2. **Probabilistic setting:**

$p(1), p(2), \dots p(r) \in \mathbf{P} \sim$ IID according to $\tilde{\mathbf{Q}}$

$$\tilde{\mathbf{Q}} \triangleq \begin{bmatrix}\tilde{Q}(1) & \cdots & \tilde{Q}(S)\end{bmatrix}, \qquad \mathbf{P} \triangleq \begin{bmatrix}P(1) & \cdots & P(S)\end{bmatrix}$$

Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **IV:** $\vec{V} = 1/\sum_{i=1}^{S} \frac{\tilde{Q}(i)}{1-P(i)}$ (as in fixed-channels type setting)
- **EE, 1st expression:** for $\alpha < \vec{V}$: $E^{\text{prob}}(\tilde{\mathbf{Q}}) = (1-\alpha)\log x - \alpha \sum_{i=1}^{S} \tilde{Q}(i) \log \frac{1-P(i)}{1-P(i)x}$

  where $x \in (1, 1/\max \mathbf{P})$ is the solution of $\sum_{i=1}^{S} \tilde{Q}(i) \cdot \{1-P(i)\} \cdot \frac{1-\alpha-P(i)x}{\{1-P(i)x\}^2} = 0$

Corollary: For $r = o(N)$, EE equals that of worst link: $E = -\log \max \mathbf{P}$.

Single Packet, Heterogeneous Links: Probabilistic Setting

- Each τ_ℓ is now a mixture of geometric distributions:

$$\Pr(\tau = \ell) = \sum_{i=1}^{S} \tilde{Q}(i) \cdot \{P(i)\}^{\ell-1} \cdot \{1 - P(i)\}, \qquad \mathbb{E}[\tau] = \sum_{i=1}^{S} \frac{\tilde{Q}(i)}{1 - P(i)}$$

- $\{\tau_\ell\}$ are IID

Proof of IV characterization: Repeat proof steps for homogeneous links:

$$\lim_{N\to\infty} P_e(N) = \lim_{N\to\infty} \Pr\left(\frac{1}{r}\sum_{i=1}^{r} \tau_i > \frac{N}{r}\right) = \begin{cases} 0, & \mathbb{E}[\tau] < 1/\alpha \\ 1, & \mathbb{E}[\tau] > 1/\alpha \end{cases}$$

with $\alpha \triangleq r/N$, yields $\vec{V} = 1/\mathbb{E}[\tau] = 1/\sum_{i=1}^{S} \frac{\tilde{Q}(i)}{1-P(i)}$

Single Packet, Heterogeneous Links: Fixed Channels-Type

Proof of (1st) EE expression: By Cramér's theorem:

$$E = -\sup_{\nu>0}\{\nu - \alpha \log M_\tau(\nu)\}$$

- M_τ is the moment-generating function of τ
- *Chernoff's upper bound on P_e:* Results in the above EE (achievable only)

Single Packet, Heterogeneous Links: Fixed Channels-Type

Theorem [Domanovitz–Philosof–K. INFOCOM'22]:

- **IV:** $\vec{V} = 1/\sum_{i=1}^{S} \frac{\tilde{Q}(i)}{1-P(i)}$
- **EE, 1st expression:** for $\alpha < \vec{V}$: $E^{\text{prob}}(\tilde{\mathbf{Q}}) = (1-\alpha)\log x - \alpha \sum_{i=1}^{S} \tilde{Q}(i) \log \frac{1-P(i)}{1-P(i)x}$

  where $x \in (1, 1/\max \mathbf{P})$ is the solution of $\sum_{i=1}^{S} \tilde{Q}(i) \cdot \{1-P(i)\} \cdot \frac{1-\alpha-P(i)x}{\{1-P(i)x\}^2} = 0$

- **EE, 2nd expression:** for $\alpha < \vec{V}$: $E^{\text{prob}}(\tilde{\mathbf{Q}}) = \min_{\mathbf{Q} \in \Delta_{S-1}} \left\{E^{\text{fixed}}(\mathbf{Q}) + \alpha \mathbb{D}(\mathbf{Q}||\tilde{\mathbf{Q}})\right\}$

  where $\Delta_{S-1} \triangleq \left\{(x_1, \ldots, x_S) \middle| \sum_{i=1}^{S} x_i = 1,\ x_i \geq 0\ \forall i\right\}$ is the standard simplex

Remark: Both EE expressions can be proved to be equal also directly via the KKT conditions.

Single Packet, Heterogeneous Links: Fixed Channels-Type

Proof of 2nd EE expression:

$$P_e(N) = \Pr\left(\sum_{i=1}^{r} \tau_i > N\right) = \sum_{\mathbf{Q}\in\mathcal{Q}_N} \Pr\left(\sum_{i=1}^{r} \tau_i > N \middle| \mathbf{Q}_p = \mathbf{Q}\right) \Pr\left(\mathbf{Q}_p = \mathbf{Q}\right)$$

$$\doteq \max_{\mathbf{Q}\in\mathcal{Q}_N} \exp\left\{-N \cdot E^{\text{fixed}}(\mathbf{Q})\right\} \cdot \exp\left\{-r \cdot \mathbb{D}\left(\mathbf{Q}||\widetilde{\mathbf{Q}}\right)\right\}$$

$$= \exp\left\{-N \cdot \min_{\mathbf{Q}\in\mathcal{Q}_N}\left\{E^{\text{fixed}}(\mathbf{Q}) + \alpha\mathbb{D}\left(\mathbf{Q}||\widetilde{\mathbf{Q}}\right)\right\}\right\}$$

- $\mathcal{Q}_N \triangleq$ All partitions of N balls into S cells, divided by N (the “type class”)

Fixed Channels-Type Setting vs. Probabilistic Setting

$\mathbf{Q} = \tilde{\mathbf{Q}} = (0.5 \quad 0.2 \quad 0.3)$, $\mathbf{P} = (0.2 \quad 0.5 \quad 0.7)$

- $\vec{V} = 0.5$