Joint Source–Channel Coding Under an Input Energy Constraint

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JSCC Over AWGN With Unlimited Bandwidth

**Source:** $x^k \in \mathbb{R}^k$—i.i.d. entries with density $f_x(a)$

**Transmitter:** $s_{x^k}(t)$
- Input energy constraint $E$: $\int_{-kT/2}^{kT/2} |s_{x^k}(t)|^2 \, dt \leq kE, \quad \forall x^k$

**Channel:** $r(t) = s_{x^k}(t) + n(t)$
- $n$ continuous-time AWGN with two-sided spectral density $N/2$

**Power limited regime:** $C = \frac{PT}{N} = \frac{E}{N} \triangleq$ ENR

What is the minimum distortion $D \triangleq \frac{1}{k} \mathbb{E} \left[ \|x^k - \hat{x}^k\|^2 \right]$?
JSCC Over AWGN With Unlimited Bandwidth

<table>
<thead>
<tr>
<th></th>
<th>Infinite blocklength</th>
<th>Finite blocklength</th>
</tr>
</thead>
<tbody>
<tr>
<td>Known ENR</td>
<td>[Shannon '59]</td>
<td>[Burnashev '84, '86]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[Sevinç and Tuncel '16]</td>
</tr>
<tr>
<td>Unknown ENR</td>
<td>[Koken–Tuncel '17]</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>[Baniasadi–Tuncel '20]</td>
<td></td>
</tr>
</tbody>
</table>

**Known ENR**
- Both the transmitter and the receiver know the exact ENR
- The signal $s_{x^k}$ may be optimized for the true ENR value
- This work: scalar source ($k = 1$)

**Unknown ENR**
- Only the receiver knows the ENR
- The transmitter accommodate for multiple noise levels
Part I

Known ENR
Infinite Blocklength \( (k \to \infty) \)

### Outer Bound — Separation [Shannon ’59]

- The optimal distortion:

\[
h(x) - \frac{1}{2} \log(2\pi e D) \leq R(D) \leq C = WT \cdot \log \left(1 + \frac{P}{N \cdot W}\right)
\]

\[
\lim_{W \to \infty} WT \cdot \log \left(1 + \frac{P}{N \cdot W}\right) = \frac{PT}{N} \triangleq \text{ENR}
\]

\[
D_\infty^* \geq \frac{e^{2h(x)}}{2\pi e} e^{-2\text{ENR}}
\]

- Tight for \( k \to \infty \), Gaussian source and known ENR

- Cannot be attained for finite blocklength [Gastpar ’03]

### Achievability — Random Coding

- Separation-based random coding

- Achieves \( D_\infty^* \) for \( k \to \infty \)
Scalar Source \((k = 1)\)

Scalar Source Bound \([\text{Burnashev '84, '86}]\)

- For \(k = 1\):
  \[
  D^* \geq K_1 \cdot e^{-\frac{\text{ENR}}{3}} \cdot \text{ENR}^{-K_2} \cdot \{1 + o(1)\}
  \]
- Exact constants \(K_1\) and \(K_2\) are unknown
- **Exponentially tight**

Scalar Source — Linear Transmission

- Linear transmission, receiver employs LMMSE
  
  \[
  s_x(t) = \sqrt{E} \frac{x}{\sigma_x} \varphi(t),
  \]
  \[
  y = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) r(t) dt,
  \]
  \[
  \hat{x} = \alpha_{\text{MMSE}} \cdot y
  \]
- The distortion is bounded by \(\frac{\sigma_x^2}{1+2\text{ENR}}\)
  - Decays **linearly** instead of **exponentially**

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JSCC Under an Input Energy Constraint
Scalar Source ($k = 1$)

**Closing the Gap — Channel Coding [Burnashev ’84, ’86]**

1. Quantize the source using uniform scalar quantization
2. Transmit with orthogonal signaling (capacity achieving)
   - $s_i(t)$ is the $i$'th basis function of an orthogonal basis of size $N$

**Scalar quantization—orthogonal modulation scheme:**

![Diagram of scalar quantization and orthogonal modulation](image)

- Quantization ($Q()$)
- Orthogonal Signaling
- Additive White Gaussian Noise ($n(t)$)
- Transmission ($s_i(t)$)
- Additive White Gaussian Noise ($r(t)$)
- Maximum Likelihood Decoding ($\hat{i}$)
- Dequantization ($Q^{-1}()$)
- Estimate ($\hat{x}$)

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Figure: $x = 0.11$, $i = 0$

Figure: $x = 0.3$, $i = 1$
Known ENR – Achievability Bounds

Performance
- Distortion is upper bounded by:
  \[ D \leq K \cdot e^{-\frac{\text{ENR}}{3}} \cdot \{1 + o(1)\} \]
- Achieves the exponential rate of the outer bound
- \textbf{Is } K_2 = 0 \textbf{ optimal?}
- The constant } K \text{ can be optimized by the usage of non-uniform quantizer [Sevinç and Tuncel ’16]

Disadvantages
- \textbf{Separation:}
  1. Fails to attain the next decaying order of } \text{ENR}^{-K_2}
  2. Saturates for high ENRs
- \textbf{Scalar quantization is sub-optimal!}
  - Quantization \rightarrow \text{finite set of channel inputs}
  - Close values mapped to same quantization bin
    (transmitter-side “hard-decision”)
  - Correct channel decoding—quantization dictates performance
Known ENR – Achievability Bounds

Idea: JSCC instead of quantization and channel coding
Analog PPM

- Transmit a rectangular pulse with width $1/\beta$
- Pulse delay = current source realization
- No quantization — fully analog transmission
- Estimate delay using Maximum A-Posteriori (MAP) estimator

JSCC using analog PPM scheme:

$$\lambda(a) = \frac{2\sqrt{E}}{N} \int_{-\infty}^{\infty} r(t) \phi(t - a\Delta) dt + \log(f_x(a))$$

- $f_x(a)$ — the source density
- $\phi$ can be optimized—rectangular pulse allows analytical analysis
Analog PPM

**JSCC using analog PPM scheme:**

\[
x \rightarrow \text{Transmitter} \rightarrow \sqrt{E_{\phi}(t - x \Delta)} \rightarrow \text{Receiver} \rightarrow r(t) = \hat{x} = \arg\max_{a \in \mathbb{R}} \{\lambda(a)\}
\]

- **Key properties**
  - No quantization — direct mapping of source to channel inputs
  - Distinction between close source values (transmitter-side “soft-decision”)
  - Small noise values — small distortion

**Schematic diagram**

- Transmitter: \( x \)
- Receiver: \( \hat{x} \)
- Noise: \( n(t) \)
- Output: \( r(t) \)

**Graphs**

- 0.5 - 0.5
- 0 - 0.5
- 0.05
- 0.1
- 0.15
- 0.2
- 0.25
- 0.3
- 0.35
- 0.4
- 0.45
- 0.5

**Equations**

- Transmitter: \( \sqrt{E_{\phi}(t - x \Delta)} \)
- Receiver: \( \hat{x} = \arg\max_{a \in \mathbb{R}} \{\lambda(a)\} \)
Analog PPM hinges on delay estimation of known pulse corrupted by AWGN.

1. Fundamental problem in signal processing and radar
2. Received much attention over the years — closed-form distortion expressions remain an open problem

Analysis

Analysis based on properties of Wiener processes:

1. Split to large and small errors:

   \[ E[\epsilon^2] \leq E[\epsilon^2| |\epsilon| \leq \frac{1}{\beta}] + P\left(|\epsilon| > \frac{1}{\beta}\right)E[\epsilon^2| |\epsilon| > \frac{1}{\beta}] \]

   \[ \triangleq D_S + P_LD_L \]

2. Small errors: second moment of the argmax of a Wiener process with drift [Zehavi '84]
3. Large errors: probability of error for orthogonal signals with unequal prior [Ziv and Zakai '69]
Proposition: Uniform Source Performance

\[ D \leq D_S + D_L P_L \]

- \( D_S \triangleq \frac{13/8}{(\beta \text{ENR})^2} \cdot \{1 + o(1)\} \)
- \( D_L \triangleq \frac{1}{6} \left( 1 + \frac{2}{\beta} + \frac{4}{\beta^2} \right) \)
- \( P_L \triangleq \frac{\beta \sqrt{\text{ENR}} e^{-\frac{\text{ENR}}{2}}}{16 \sqrt{\pi}} \cdot \{1 + o(1)\} \)

Theorem: Upper Bound on Optimal Distortion

Setting \( \beta = (312 \sqrt{\pi})^{\frac{1}{3}} (\text{ENR})^{-\frac{5}{6}} e^{\frac{\text{ENR}}{6}} \) yields

\[ D \leq 0.072 e^{-\frac{\text{ENR}}{3}} \cdot \text{ENR}^{-\frac{1}{3}} \cdot \{1 + o(1)\} \]
Proposition: Gaussian Source Performance

\[ D \leq D_S + D_L \]

- \( D_S \triangleq \frac{13/8}{(\beta \text{ENR})^2} \cdot \{1 + o(1)\} \)
- \( D_L \triangleq 2\beta \sqrt{\text{ENR}} \cdot e^{-\frac{\text{ENR}}{2}} \cdot \{1 + o(1)\} \)

Theorem: Upper Bound on Optimal Distortion

Setting \( \beta = \left( \frac{13}{8} \right)^{\frac{1}{3}} (\text{ENR})^{-\frac{5}{6}} e^{\frac{\text{ENR}}{6}} \) yields

\[ D \leq 3 \cdot \left( \frac{13}{8} \right)^{\frac{1}{3}} e^{-\frac{\text{ENR}}{3}} \cdot \text{ENR}^{-\frac{1}{3}} \cdot \{1 + o(1)\} \]
Numerical Evaluation

- Numerical optimization of the empirical performance and the complete bounds
- Comparison to Burnashev and Sevinç–Tuncel performance

Figure: Gaussian source SDR

Figure: Uniform source SDR
Analog PPM—Robustness

“Why Analog PPM is good for Unknown ENR”

ENR Robustness

For $\text{ENR} > \text{ENR}_0$, the term $D_L P_L$ is negligible:

$$D \simeq \frac{13/8}{(\beta \text{ENR})^2}$$

$\text{ENR} > \text{ENR}_0$ — quadratic decay

Figure: Gaussian source, $\beta = 3.68$
Known ENR Summary

Summary

- Separation based schemes:
  1. Does not achieve additional polynomial decay
  2. Does not improve for $\text{ENR} > \text{ENR}_0$

- We circumvent those drawbacks by using analog JSCC
  1. Achieves both exponential and polynomial decay terms—\textbf{Settles an open problem about the gap between Burnashev’s bounds}
  2. Improves quadratically for $\text{ENR} > \text{ENR}_0$

Limitations

- $\beta$ can be optimized only for specific ENR
- For constant $\beta$, the distortion decays quadratically
Part II

Unknown ENR
Reminder

- Now, only the receiver knows the noise level $N$
- The transmitter accommodate for multiple noise levels

**Main goal:** minimum energy to achieve

$$D \leq \sigma^2_x \mathcal{F}(N), \forall N \geq 0$$

We concentrate on polynomial profiles:

$$\mathcal{F}(N) = \frac{1}{1 + \left( \frac{\tilde{E}}{N} \right)^L}, \tilde{E} \in \mathbb{R}^+$$

- Scenario received much attention in the finite bandwidth case
  [Mittal–Phamdo ‘02], [Reznic–Feder–Zamir ‘06],
  [Santhi–Vardy ‘06], [Bhattad–Narayanan ’10]
- Results for the infinite bandwidth scenario attained recently
  [Köken–Tuncel ’17], [Baniasadi–Tuncel ’20]
Outer Bounds

Polynomially Decay Profile

- Separation based bound [Baniasadi–Tuncel ’20]:
  \[ E_{\text{min}} (L = 2) \geq 0.804 \tilde{E} \]

- Improved family of bounds: utilize the connection between robust transmission and the broadcast channel
  [Reznic–Feder–Zamir ’06], [Köken–Tuncel ’17],
  [Baniasadi–Tuncel ’20]

- Best known bound [Baniasadi–Tuncel ’20]:
  \[ E \geq E_{\text{min}} (L = 2) \approx 0.905 \tilde{E} \]

Exponentially Decay Profile [Köken–Tuncel ’17]

- Exponentially decay profile: \[ E_{\text{min}} \to \infty \]
Achievability: Hybrid Digital–Analog Schemes

Layered Transmission AWGN With $W \rightarrow \infty$ [Köken–Tuncel ’17]

- **Transmitter:**
  - Uncoded transmission: $Y_0 = \sqrt{E_0}X + N_0$
  - Digital transmission: $S_k = Q(e_k), e_k = S_{k-1} - Q(S_{k-1})$

- **Receiver:**
  1. Digital: Decodes $\{S_k\}_{k=1}^{\ell}, \ell$ is determined from noise level
  2. Analog: Estimates $\hat{e}_\ell$ from $\{\hat{S}_k\}_{k=1}^{\ell}$ and $Y_0$, to generate $\hat{X}$

- Total energy: $E_{\text{min}} \approx 3.184\tilde{E}$

Multiple Uncoded Transmissions [Baniasadi–Tuncel ’20]

- Combining multiple uncoded transmissions
- Minimal achievable energy: $E_{\text{min}} \approx 2.32\tilde{E}$
Achievability: Hybrid Digital–Analog Schemes

Drawbacks

- Performance saturation:
  - Quantization does not improve with ENR
  - Uncoded transmissions—improve only linearly
- Linear transmission—inefficient use of BW

Finite BW: Modulo-lattice modulation (MLM) [Reznic–Feder–Zamir ’06]
Modulo-based Wyner–Ziv coding outperforms success refinement.

Our Approach: Two improvements

1. Use MLM in infinite BW setting:
   All layers designed for ENR < True ENR, improve with ENR

2. Replace linear transmissions with analog PPM:
   Better utilization of BW
Modulo Lattice Modulation (MLM)

- Source comprises of known \( (j^k) \) and unknown \( (q^k) \) parts
  \[ x^k = q^k + j^k \]

  Transmitted over an additive channel:
  \[ y^k = m^k \left( x^k \right) + z^k, \quad \frac{1}{k} \mathbb{E} \left[ \| m^k \|^2 \right] \leq \sigma_z^2 \]

- Transmitter:
  \[ m^k = [\eta x^k + d^k]_{\Lambda} \]
  \[ d^k \sim \text{Unif} \left( \mathcal{V}_0 \right), \text{ dither signal} \]

- Receiver:
  \[ \tilde{y}^k = [\alpha_c y^k - \eta j^k - d^k]_{\Lambda} = [\eta q^k + z_{\text{eff}}^k]_{\Lambda}, \]
  \[ \hat{x}^k = \frac{\alpha_s}{\eta} \tilde{y}^k + j^k \]
Modulo-Lattice Modulation (MLM)

MLM Performance [Kochman Zamir '09, Ordentlich Erez '16]

Let $q^k$ and $z^k$ be semi norm-ergodic sequences. Then,

$$D \leq \frac{\sigma_q^2}{1 + \text{SNR}} \triangleq D^*$$

- Semi norm-ergodic:
  $$\Pr \left( a^k \notin \mathcal{B} \left( 0, \sqrt{(1 + \delta) k\sigma_a} \right) \right) \leq \epsilon$$

Universality [Kochman–Zamir '09, Ordentlich–Erez '16]

- SNR $\geq$ SNR$_0$, $\sigma_q^2 \leq \tilde{\sigma}_q^2$
- Tx oblivious of SNR and $\sigma_q^2$, knows SNR$_0$ and $\tilde{\sigma}_q^2$

$$D \leq \min \left\{ \frac{\sigma_q^2}{1 + \text{SNR}_0}, \frac{\tilde{\sigma}_q^2}{1 + \text{SNR}} \cdot \frac{1 + \text{SNR}_0}{\text{SNR}_0} \right\}$$

- For SNR $\geq$ SNR$_0 \gg 1$: $D \to \frac{\sigma_q^2}{1 + \text{SNR}} = D^*$
**M-Layer Transmitter**

“Black-box” approach:
- Generate set of “layers” using modulo-encoding (MLM)
- Transmit independently using scalar JSCC scheme

M Layer Transmitter:

- **MLM Tx**
- **JSCC Encoding**
- **JSCC Estimation**
- **MLM Rx**

Equivalent Channel:
- MLM operates over the channel $m_i^k \rightarrow \hat{m}_i^k$
  1. Equivalent noise is JSCC output noise
  2. Noise not necessarily Gaussian
M-Layer Transmitter

**Transmitter:**
- **MLM part:** Generates the set of signals
  \[ m_i^k = [\eta_i x^k + d_i^k]_{\Lambda}, \quad i = 1, \ldots, M \]
- **JSCC part:**
  1. Transmits the entries of \(x^k\) using linear transmission
  2. Transmits the entries of \(m_i^k\) using scalar JSCC scheme with \(E_i\)

**Receiver:** For \(i = 1, \ldots, M\)
- **JSCC part:** Generates \(\hat{m}_i^k\) using the JSCC receiver
- **MLM part:** Uses the MLM receiver over the channel
  \(m_i^k \rightarrow \hat{m}_i^k\) with \(\hat{x}_{i-1}^k\) as SI, to generate \(\hat{x}_i^k\)
Interleaving

- Interleaving is used between MLM and JSCC steps
- Receiver applies de-interleaving between JSCC and MLM
- Ensures independence between vector entries
- Allows to use MLM results for semi norm-ergodic variables

Gaussianization [AMIMON ’07, No–Weissman ’16, Erez–Hadad ’16]

- PPM inputs multiplied by orthogonal matrix
- Allows to use analysis of JSCC with Gaussian inputs
- Receiver multiplies by $H^{-1} = H^T$—Gaussianizes the noise

- Dimension $k$ is general—closed form solutions only for $k \to \infty$
Let $L > 1$, $\tilde{E} > 0$ and $k \to \infty$. Then, polynomial profile is achievable for any transmit energy $E$ that satisfies

$$E > \delta_{\text{lin}}(L) \tilde{E},$$

where

$$\delta_{\text{lin}}(L) = \frac{1}{2} \cdot \min_{(\alpha,x) \in \mathbb{R}^2_+} \left\{ \left( \frac{e^{\alpha}}{x} \right)^{L-1} + \frac{x}{2} \left( e^{\alpha L} - 1 \right) \left( 1 + \sqrt{1 + \frac{4e^{\alpha(L+1)}}{(1 - e^{\alpha L})^2}} \right) \frac{e^{-2\alpha}}{1 - e^{-\alpha}} \right\}.$$

- For $L = 2$, we get $E > 2.167\tilde{E}$
- Better than $2.32\tilde{E}$ of other schemes
- Achieved only by MLM (no PPM yet)
M-Layer Transmitter: PPM Transmissions

Let $L = 2$, $\tilde{E} > 0$ and $k \to \infty$. Then, polynomial profile is achievable for any transmit energy $E$ that satisfies $E > 1.961\tilde{E}$.

- Offers further improvement over linear-based transmitter
- PPM bounds are not tight—improvement is even higher

**Figure: Distortion**

**Figure: Accumulated Energy**

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JSCC Under an Input Energy Constraint
M-Layer Transmitter: Scalar

- Numerical simulation of uniform source with $k = 1$
- Quadratic profile

Scheme applies for practical low-delay scenario
- PPM allows saving of transmitted energy

Figure: Distortion

Figure: Accumulated Energy
We introduced robust energy-efficient communication scheme

1. Based on MLM and JSCC
2. Analog PPM offers performance boost

Total energy is lower than best reported results
- Real improvement is even higher than theoretical bounds

Scheme can be used in practical low-delay scenarios
Future Research

**Known ENR**
- Optimal polynomial decay:
  - Burnashev’s bound: $K_1 = K_2$
  - MAP estimator is sub-optimal [Ibragimov–Khas’minskii ’75]: Analysis of full MMSE decoder
- Vector sources: multidimensional mappings

**Unknown ENR**
- Tightening bounds:
  - Analysis of inner-bound can be improved
  - Different line of works: outer bound
- Universal SI at the receiver
- Dual problem: near-zero bandwidth [Baniasadi–Tuncel ’20]
Questions?