Joint Source–Channel Coding Under an Input Energy Constraint

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JSCC Over AWGN With Unlimited Bandwidth

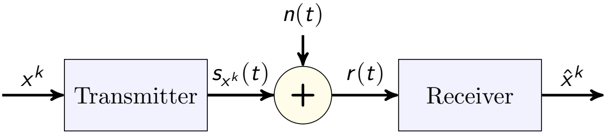
Source: $x^k \in \mathbb{R}^k$—i.i.d. entries with density $f_x(a)$

Transmitter: $s_{x^k}(t)$

- Input energy constraint E: $\int_{-\frac{kT}{2}}^{\frac{kT}{2}} \left|s_{x^k}(t)\right|^2 dt \leqslant kE, \quad \forall x^k$

Channel: $r(t) = s_{x^k}(t) + n(t)$

- n continuous-time AWGN with two-sided spectral density $N/2$



- Power limited regime: $C = \frac{PT}{N} = \frac{E}{N} \triangleq \mathrm{ENR}$

What is the minimum distortion $D \triangleq \frac{1}{k}\mathbb{E}\left[\left\|x^k - \hat{x}^k\right\|^2\right]$?

JSCC Over AWGN With Unlimited Bandwidth

	Infinite blocklength	Finite blocklength
Known ENR	[Shannon '59]	[Burnashev '84, '86] [Sevinç and Tuncel '16]
Unknown ENR	[Koken–Tuncel '17] [Baniasadi–Tuncel '20]	?

Known ENR

- Both the transmitter and the receiver know the exact ENR
- The signal s_{x^k} may be optimized for the true ENR value
- This work: scalar source ($k = 1$)

Unknown ENR

- Only the receiver knows the ENR
- The transmitter accommodate for multiple noise levels

Part I

Known ENR

Infinite Blocklength ($k \to \infty$)

Outer Bound — Separation [Shannon '59]

- The optimal distortion:

$$h(x) - \frac{1}{2}\log(2\pi e D) \leqslant R(D) \leqslant C = WT \cdot \log\left(1 + \frac{P}{N \cdot W}\right)$$

  - $\lim_{W\to\infty} WT \cdot \log\left(1 + \frac{P}{N \cdot W}\right) = \frac{PT}{N} \triangleq \mathrm{ENR}$

$$D_\infty^* \geqslant \frac{e^{2h(x)}}{2\pi e} e^{-2\mathrm{ENR}}$$

- Tight for $k \to \infty$, Gaussian source and known ENR
- Cannot be attained for finite blocklength [Gastpar '03]

Achievability — Random Coding

- Separation-based random coding
- Achieves D_∞^* for $k \to \infty$

Scalar Source ($k = 1$)

Scalar Source Bound [Burnashev '84, '86]

- For $k = 1$:

$$D^* \geqslant K_1 \cdot \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \mathrm{ENR}^{-K_2} \cdot \{1 + o(1)\}$$

- Exact constants K_1 and K_2 are unknown
- **Exponentially tight**

Scalar Source — Linear Transmission

- Linear transmission, receiver employs LMMSE

$$s_x(t) = \sqrt{E}\frac{x}{\sigma_x}\varphi(t),$$

$$y = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) r(t) dt,$$

$$\hat{x} = \alpha_{\mathrm{MMSE}} \cdot y$$

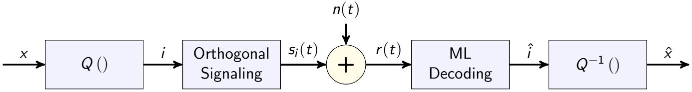
- The distortion is bounded by $\frac{\sigma_x^2}{1+2\mathrm{ENR}}$
  - Decays **linearly** instead of **exponentially**

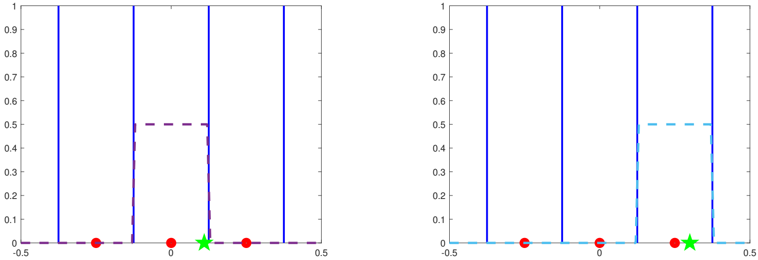
Scalar Source ($k = 1$)

Closing the Gap — Channel Coding [Burnashev '84, '86]

1. Quantize the source using uniform scalar quantization
2. Transmit with orthogonal signaling (capacity achieving)
   - $s_i(t)$ is the i'th basis function of an orthogonal basis of size N

Scalar quantization–orthogonal modulation scheme:





Known ENR – Achievability Bounds

Performance

- Distortion is upper bounded by:
$$D \leqslant K \cdot \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \{1 + o(1)\}$$
- Achieves the exponential rate of the outer bound
  - **Is $K_2 = 0$ optimal?**
- The constant K can be optimized by the usage of non-uniform quantizer [Sevinç and Tuncel '16]

Disadvantages

- Separation:
  1. Fails to attain the next decaying order of ENR^{-K_2}
  2. Saturates for high ENRs
- **Scalar quantization is sub-optimal!**
  - Quantization $\longrightarrow$ finite set of channel inputs
  - Close values mapped to same quantization bin (transmitter-side "hard-decision")
  - Correct channel decoding—quantization dictates performance

Known ENR – Achievability Bounds

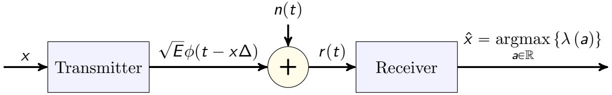
Idea: JSCC instead of quantization and channel coding

Scheme

Analog PPM

- Transmit a rectangular pulse with width $1/\beta$
- Pulse delay = current source realization
  - No quantization — fully analog transmission
- Estimate delay using Maximum A-Posteriori (MAP) estimator

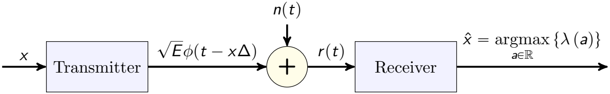
JSCC using analog PPM scheme:

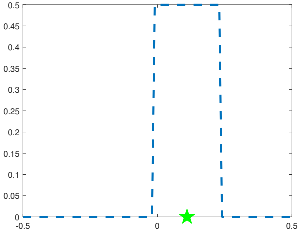


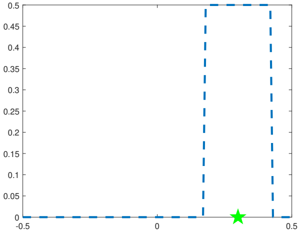
- $\lambda(a) = \frac{2\sqrt{E}}{N} \int_{-\infty}^{\infty} r(t)\phi(t - a\Delta)dt + \log(f_x(a))$
- $f_x(a)$ — the source density
- ϕ can be optimized—rectangular pulse allows analytical analysis

Analog PPM

JSCC using analog PPM scheme:







Key properties

- No quantization — direct mapping of source to channel inputs
- Distinction between close source values (transmitter-side "soft-decision")
- Small noise values — small distortion

Analog PPM

Delay Estimation

- Analog PPM hinges on delay estimation of known pulse corrupted by AWGN
  1. Fundamental problem in signal processing and radar
  2. Received much attention over the years — **closed-form distortion expressions remain an open problem**

Analysis

- Analysis based on properties of Wiener processes:
  1. Split to large and small errors:
  $$\mathbb{E}\left[\epsilon^2\right] \leqslant \mathbb{E}\left[\epsilon^2 \middle| |\epsilon| \leqslant \frac{1}{\beta}\right] + P\left(|\epsilon| > \frac{1}{\beta}\right)\mathbb{E}\left[\epsilon^2 \middle| |\epsilon| > \frac{1}{\beta}\right]$$
  $$\triangleq D_S + P_L D_L$$
  2. Small errors: second moment of the argmax of a Wiener process with drift [Zehavi '84]
  3. Large errors: probability of error for orthogonal signals with unequal prior [Ziv and Zakai '69]

Analog PPM—Uniform Source

Proposition: Uniform Source Performance

$$D \leqslant D_S + D_L P_L$$

- $D_S \triangleq \frac{13/8}{(\beta \mathrm{ENR})^2} \cdot \{1 + o(1)\}$
- $D_L \triangleq \frac{1}{6}\left(1 + \frac{2}{\beta} + \frac{4}{\beta^2}\right)$
- $P_L \triangleq \frac{\beta\sqrt{\mathrm{ENR}}\mathrm{e}^{-\frac{\mathrm{ENR}}{2}}}{16\sqrt{\pi}} \cdot \{1 + o(1)\}$

Theorem: Upper Bound on Optimal Distortion

- Setting $\beta = (312\sqrt{\pi})^{\frac{1}{3}} (\mathrm{ENR})^{-\frac{5}{6}} \mathrm{e}^{\frac{\mathrm{ENR}}{6}}$ yields

$$D \leqslant 0.072\, \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \mathrm{ENR}^{-\frac{1}{3}} \cdot \{1 + o(1)\}$$

Analog PPM—Gaussian Source

Proposition: Gaussian Source Performance

$$D \leqslant D_S + D_L$$

- $D_S \triangleq \frac{13/8}{(\beta \mathrm{ENR})^2} \cdot \{1 + o(1)\}$
- $D_L \triangleq 2\beta\sqrt{\mathrm{ENR}} \cdot \mathrm{e}^{-\frac{\mathrm{ENR}}{2}} \cdot \{1 + o(1)\}$

Theorem: Upper Bound on Optimal Distortion

- Setting $\beta = \left(\frac{13}{8}\right)^{\frac{1}{3}} (\mathrm{ENR})^{-\frac{5}{6}} \mathrm{e}^{\frac{\mathrm{ENR}}{6}}$ yields

$$D \leqslant 3 \cdot \left(\frac{13}{8}\right)^{\frac{1}{3}} \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \mathrm{ENR}^{-\frac{1}{3}} \cdot \{1 + o(1)\}$$

Numerical Evaluation

- Numerical optimization of the empirical performance and the complete bounds
- Comparison to Burnashev and Sevinç–Tuncel performance

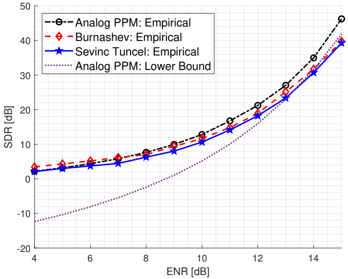


Figure: Gaussian source SDR

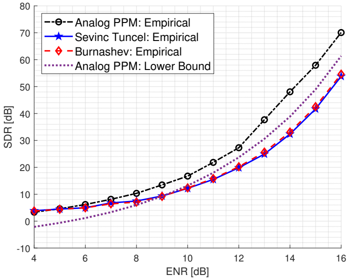


Figure: Uniform source SDR

Analog PPM—Robustness

"Why Analog PPM is good for Unknown ENR"

ENR Robustness

For $\mathrm{ENR} > \mathrm{ENR}_0$, the term $D_L P_L$ is negligible:

$$D \cong \frac{13/8}{(\beta \mathrm{ENR})^2}$$

- $\mathrm{ENR} > \mathrm{ENR}_0$ — **quadratic decay**

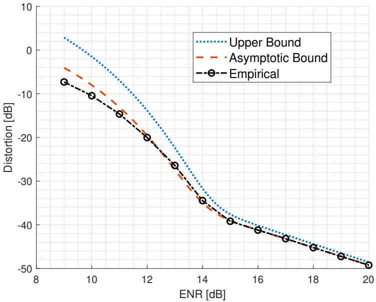


Figure: Gaussian source, $\beta = 3.68$

Known ENR Summary

Summary

- Separation based schemes:
  1. Does not achieve additional polynomial decay
  2. Does not improve for $\mathrm{ENR} > \mathrm{ENR}_0$
- We circumvent those drawbacks by using analog JSCC
  1. Achieves both exponential and *polynomial* decay terms—**Settles an open problem about the gap between Burnashev's bounds**
  2. Improves quadratically for $\mathrm{ENR} > \mathrm{ENR}_0$

Limitations

- β can be optimized only for specific ENR
- For constant β, the distortion decays quadratically

Part II

Unknown ENR

Unknown ENR Regime

Reminder

- Now, only the receiver knows the noise level N
- The transmitter accommodate for multiple noise levels
- **Main goal**: minimum energy to achieve

$$D \leqslant \sigma_x^2 \mathscr{F}\left(N\right), \forall N \geqslant 0$$

We concentrate on polynomial profiles:

$$\mathscr{F}\left(N\right) \triangleq \frac{1}{1+\left(\frac{\tilde{E}}{N}\right)^L}, \tilde{E} \in \mathbb{R}^+$$

- Scenario received much attention in the finite bandwidth case [Mittal–Phamdo '02], [Reznic–Feder–Zamir '06], [Santhi–Vardy '06], [Bhattad–Narayanan '10]
- Results for the infinite bandwidth scenario attained recently [Köken–Tuncel '17], [Baniasadi–Tuncel '20]

Outer Bounds

Polynomially Decay Profile

- Separation based bound [Baniasadi–Tuncel '20]:

$$E_{\min}\left(L=2\right) \geqslant 0.804\tilde{E}$$

- Improved family of bounds: utilize the connection between robust transmission and the broadcast channel [Reznic–Feder–Zamir '06], [Köken–Tuncel '17], [Baniasadi–Tuncel '20]
- Best known bound [Baniasadi–Tuncel '20]:

$$E \geqslant E_{\min}\left(L=2\right) \cong 0.905\tilde{E}$$

Exponentially Decay Profile [Köken–Tuncel '17]

- Exponentially decay profile: $E_{\min} \to \infty$

Achievability: Hybrid Digital–Analog Schemes

Layered Transmission AWGN With $W \to \infty$ [Köken–Tuncel '17]

- Transmitter:
  - Uncoded transmission: $Y_0 = \sqrt{E_0}X + N_0$
  - Digital transmission: $S_k = Q\left(e_k\right), e_k = S_{k-1} - Q\left(S_{k-1}\right)$
- Receiver:
  1. Digital: Decodes $\{S_k\}_{k=1}^{\ell}$, ℓ is determined from noise level
  2. Analog: Estimates $\hat{e}_\ell$ from $\left\{\hat{S}_k\right\}_{k=1}^{\ell}$ and Y_0, to generate $\hat{X}$
- Total energy: $E_{\min} \cong 3.184\tilde{E}$

Multiple Uncoded Transmissions [Baniasadi–Tuncel '20]

- Combining multiple uncoded transmissions
- Minimal achievable energy: $E_{\min} \cong 2.32\tilde{E}$

Achievability: Hybrid Digital–Analog Schemes

Drawbacks

- Performance saturation:
  - Quantization does not improve with ENR
  - Uncoded transmissions—improve only *linearly*
- Linear transmission—inefficient use of BW

Finite BW: Modulo-lattice modulation (MLM) [Reznic–Feder–Zamir '06]

Modulo-based Wyner–Ziv coding outperforms success. refinement.

Our Approach: Two improvements

1. Use MLM in infinite BW setting:
   All layers designed for ENR $<$ True ENR, improve with ENR
2. Replace linear transmissions with analog PPM:
   Better utilization of BW

Modulo Lattice Modulation (MLM)

MLM

- Source comprises of known (j^k) and unknown (q^k) parts

$$x^k = q^k + j^k$$

Transmitted over an additive channel:

$$y^k = m^k\left(x^k\right) + z^k, \quad \frac{\frac{1}{k}\mathbb{E}\left[\left\|m^k\right\|^2\right]}{\sigma_z^2} \triangleq \mathrm{SNR}$$

- **Transmitter:**

$$m^k = [\eta x^k + d^k]_\Lambda$$

  - $d^k \sim \mathrm{Unif}\left(\mathscr{V}_0\right)$, dither signal

- **Receiver:**

$$\tilde{y}^k = [\alpha_c y^k - \eta j^k - d^k]_\Lambda = [\eta q^k + z_{\mathrm{eff}}^k]_\Lambda,$$

$$\hat{x}^k = \frac{\alpha_s}{\eta}\tilde{y}^k + j^k$$

Modulo-Lattice Modulation (MLM)

MLM Performance [Kochman Zamir '09, Ordentlich Erez '16]

Let q^k and z^k be *semi norm-ergodic* sequences. Then,

$$D \leqslant \frac{\sigma_q^2}{1+\mathrm{SNR}} \triangleq D^*$$

- Semi norm-ergodic:

$$\Pr\left(a^k \notin \mathscr{B}\left(0, \sqrt{(1+\delta)\, k\sigma_a}\right)\right) \leqslant \epsilon$$

Universality [Kochman–Zamir '09, Ordentlich–Erez '16]

- $\mathrm{SNR} \geqslant \mathrm{SNR}_0$, $\sigma_q^2 \leqslant \tilde{\sigma}_q^2$
- Tx oblivious of SNR and σ_q^2, knows SNR_0 and $\tilde{\sigma}_q^2$

$$D \leqslant \min\left\{\frac{\sigma_q^2}{1+\mathrm{SNR}_0}, \frac{\tilde{\sigma}_q^2}{1+\mathrm{SNR}} \cdot \frac{1+\mathrm{SNR}_0}{\mathrm{SNR}_0}\right\}$$

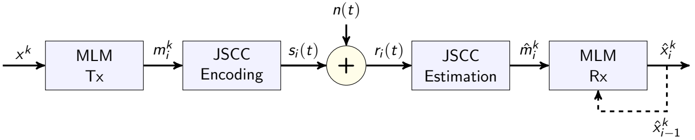
- For $\mathrm{SNR} \geqslant \mathrm{SNR}_0 \gg 1$: $D \to \frac{\sigma_q^2}{1+\mathrm{SNR}} = D^*$

M-Layer Transmitter

"Black-box" approach:

- **Generate set of "layers" using modulo-encoding (MLM)**
- **Transmit independently using scalar JSCC scheme**

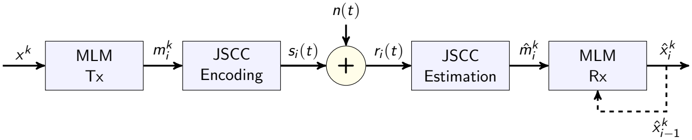
M Layer Transmitter:



Equivalent Channel

- MLM operates over the channel $m_i^k \rightarrow \hat{m}_i^k$
  1. Equivalent noise is JSCC output noise
  2. Noise not necessarily Gaussian

M-Layer Transmitter



M-Layer Transmitter

Transmitter:

- MLM part: Generates the set of signals

$$m_i^k = [\eta_i x^k + d_i^k]_\Lambda, \quad i = 1, \ldots, M$$

- JSCC part:
  1. Transmits the entries of x^k using linear transmission
  2. Transmits the entries of m_i^k using scalar JSCC scheme with E_i

Receiver: For $i = 1, \ldots, M$

- JSCC part: Generates $\hat{m}_i^k$ using the JSCC receiver
- MLM part: Uses the MLM receiver over the channel $m_i^k \to \hat{m}_i^k$ with $\hat{x}_{i-1}^k$ as SI, to generate $\hat{x}_i^k$

M-Layer Transmitter: Remarks

Interleaving

- Interleaving is used between MLM and JSCC steps
  - Receiver applies de-interleaving between JSCC and MLM
- Ensures independence between vector entries
- Allows to use MLM results for semi norm-ergodic variables

Gaussianization [AMIMON '07, No–Weissman '16, Erez–Hadad '16]

- PPM inputs multiplied by orthogonal matrix
- Allows to use analysis of JSCC with Gaussian inputs
- Receiver multiplies by $H^{-1} = H^T$—Gaussianizes the noise

- Dimension k is general—closed form solutions only for $k \to \infty$

M-Layer Transmitter: Linear

M-Layer Transmitter: Linear Transmissions

Let $L > 1, \tilde{E} > 0$ and $k \to \infty$. Then, polynomial profile is achievable for any transmit energy E that satisfies

$$E > \delta_{\mathrm{lin}}(L)\,\tilde{E},$$

where

$$\delta_{\mathrm{lin}}(L) \triangleq \frac{1}{2} \cdot \min_{(\alpha,x)\in\mathbb{R}_+^2} \left\{ \left(\frac{\mathrm{e}^{\alpha}}{x}\right)^{L-1} + \frac{x}{2}\left(\mathrm{e}^{\alpha L} - 1\right)\left(1 + \sqrt{1 + \frac{4\mathrm{e}^{\alpha(L+1)}}{\left(1-\mathrm{e}^{\alpha L}\right)^2}}\right)\frac{\mathrm{e}^{-2\alpha}}{1-\mathrm{e}^{-\alpha}} \right\}$$

- For $L = 2$, we get $E > 2.167\tilde{E}$
  1. **Better than** $2.32\tilde{E}$ **of other schemes**
  2. Achieved only by MLM (no PPM yet)

M-Layer Transmitter: PPM

M-Layer Transmitter: PPM Transmissions

Let $L = 2, \tilde{E} > 0$ and $k \to \infty$. Then, polynomial profile is achievable for any transmit energy E that satisfies

$$E > 1.961\tilde{E},$$

- Offers further improvement over linear-based transmitter
- PPM bounds are not tight—**improvement is even higher**

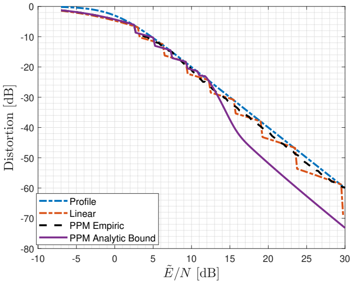


Figure: Distortion

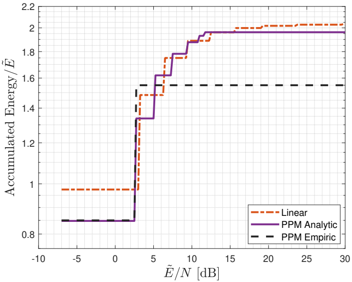


Figure: Accumulated Energy

M-Layer Transmitter: Scalar

- Numerical simulation of uniform source with $k = 1$
- Quadratic profile

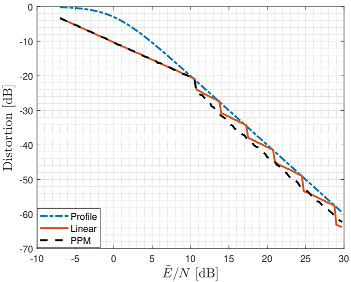


Figure: Distortion

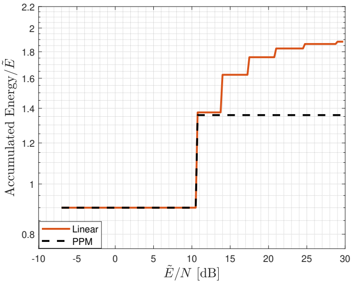


Figure: Accumulated Energy

- Scheme applies for practical low-delay scenario
  - **PPM allows saving of transmitted energy**

Unknown ENR Summary

Summary

- We introduced robust energy-efficient communication scheme
  1. Based on MLM and JSCC
  2. Analog PPM offers performance boost
- Total energy is lower than best reported results
  - Real improvement is even higher than theoretical bounds
- Scheme can be used in practical low-delay scenarios

Future Research

Known ENR

- Optimal polynomial decay:
  1. Burnashev's bound: $K_1 = K_2$
  2. MAP estimator is sub-optimal [Ibragimov–Khas'minskii '75]: Analysis of full MMSE decoder
- Vector sources: multidimensional mappings

Unknown ENR

- Tightening bounds:
  1. Analysis of inner-bound can be improved
  2. Different line of works: outer bound
- Universal SI at the receiver
- Dual problem: near-zero bandwidth [Baniasadi–Tuncel '20]

Questions?