Joint Source–Channel Coding Under an Input Energy Constraint

Omri Lev

Joint work with Anatoly Khina

Dept. EE–Systems, Tel Aviv University, Tel Aviv, Israel

IT Seminar
November, 2021

JSCC Over AWGN With Unlimited Bandwidth

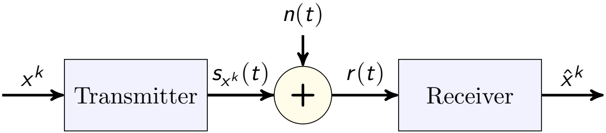
Source: $x^k \in \mathbb{R}^k$—i.i.d. entries with density $f_x(a)$

Transmitter: $s_{x^k}(t)$

- Input energy constraint E: $\int_{-\frac{kT}{2}}^{\frac{kT}{2}} \left|s_{x^k}(t)\right|^2 dt \leqslant kE, \quad \forall x^k$

Channel: $r(t) = s_{x^k}(t) + n(t)$

- n continuous-time AWGN with two-sided spectral density $N/2$



- Power limited regime: $C = \frac{PT}{N} = \frac{E}{N} \triangleq \mathrm{ENR}$

What is the minimum distortion $D \triangleq \frac{1}{k}\mathbb{E}\left[\left\|x^k - \hat{x}^k\right\|^2\right]$?

JSCC Over AWGN With Unlimited Bandwidth

	Infinite blocklength	Finite blocklength
Known ENR	[Shannon '59]	[Burnashev '84, '86] [Sevinç and Tuncel '16]
Unknown ENR	[Koken–Tuncel '17] [Baniasadi–Tuncel '20]	?

Known ENR

- Both the transmitter and the receiver know the exact ENR
- The signal s_{x^k} may be optimized for the true ENR value

Unknown ENR

- Only the receiver knows the ENR
- The transmitter accommodate for multiple noise levels

- Work has two parts:
  1. known ENR & scalar source ($k = 1$)
  2. Unknown ENR

Part I

Known ENR

Infinite Blocklength ($k \to \infty$)

Outer Bound — Separation [Shannon '59]

- The optimal distortion:

$$h(x) - \frac{1}{2}\log(2\pi e D) \leqslant R(D) \leqslant C = WT \cdot \log\left(1 + \frac{P}{N \cdot W}\right)$$

  - $\lim_{W \to \infty} WT \cdot \log\left(1 + \frac{P}{N \cdot W}\right) = \frac{PT}{N} \triangleq \mathrm{ENR}$

$$D_\infty^* \geqslant \frac{e^{2h(x)}}{2\pi e} e^{-2\mathrm{ENR}}$$

- Tight for $k \to \infty$, Gaussian source and known ENR
- Cannot be attained for finite blocklength [Gastpar '03]

Achievability — Random Coding

- Separation-based random coding
- Achieves D_∞^* for $k \to \infty$

Scalar Source ($k = 1$)

Scalar Source Bound [Burnashev '84, '86]

- For $k = 1$:

$$D^* \geqslant K_1 \cdot \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \mathrm{ENR}^{-K_2} \cdot \{1 + o(1)\}$$

- Exact constants K_1 and K_2 are unknown
- **Exponentially tight**

Scalar Source — Linear Transmission

- Linear transmission, receiver employs LMMSE

$$s_x(t) = \sqrt{E}\frac{x}{\sigma_x}\varphi(t),$$

$$y = \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) r(t) dt,$$

$$\hat{x} = \alpha_{\mathrm{MMSE}} \cdot y$$

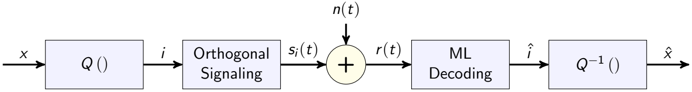
- The distortion is bounded by $\frac{\sigma_x^2}{1+2\mathrm{ENR}}$
  - Decays **linearly** instead of **exponentially**

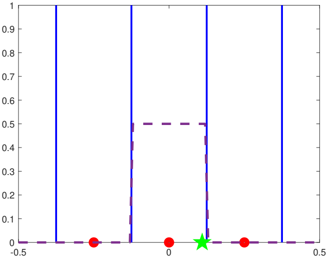
Scalar Source ($k = 1$)

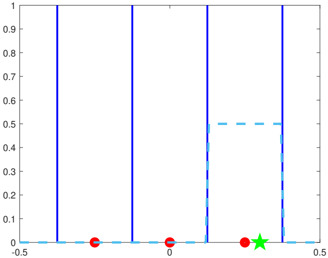
Closing the Gap — Channel Coding [Burnashev '84, '86]

1. Quantize the source using uniform scalar quantization
2. Transmit with orthogonal signaling (capacity achieving)
   - $s_i(t)$ is the i'th basis function of an orthogonal basis of size N

Scalar quantization–orthogonal modulation scheme:







Known ENR – Achievability Bounds

Performance

- Distortion is upper bounded by:

$$D \leqslant K \cdot \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \{1 + o(1)\}$$

- Achieves the exponential rate of the outer bound
  - **Is $K_2 = 0$ optimal?**
- The constant K can be optimized by the usage of non-uniform quantizer [Sevinç and Tuncel '16]

Disadvantages

- Separation:
  1. Fails to attain the next decaying order of ENR^{-K_2}
  2. Saturates for high ENRs
- **Scalar quantization is sub-optimal!**
  - Quantization $\longrightarrow$ finite set of channel inputs
  - Close values mapped to same quantization bin (transmitter-side "hard-decision")
  - Correct channel decoding—quantization dictates performance

Known ENR – Achievability Bounds

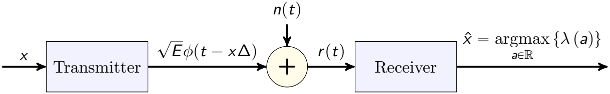
Idea: JSCC instead of quantization and channel coding

Scheme

Analog PPM

- Transmit a rectangular pulse with width $1/\beta$
- Pulse delay = current source realization
  - No quantization — fully analog transmission
- Estimate delay using Maximum A-Posteriori (MAP) estimator

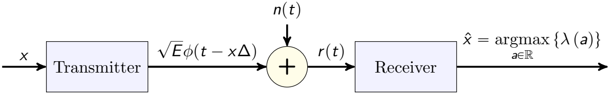
JSCC using analog PPM scheme:

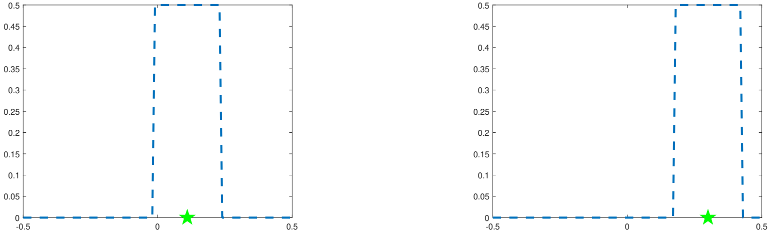


- $\lambda(a) = \frac{2\sqrt{E}}{N} \int_{-\infty}^{\infty} r(t)\phi(t - a\Delta)dt + \log(f_x(a))$
- $f_x(a)$ — the source density
- ϕ can be optimized—rectangular pulse allows analytical analysis

Analog PPM

JSCC using analog PPM scheme:





Key properties

- No quantization — direct mapping of source to channel inputs
- Distinction between close source values (transmitter-side "soft-decision")
- Small noise values — small distortion

Analog PPM

Delay Estimation

- Analog PPM hinges on delay estimation of known pulse corrupted by AWGN
  1. Fundamental problem in signal processing and radar
  2. Received much attention over the years — **closed-form distortion expressions remain an open problem**

Analysis

- Analysis based on properties of Wiener processes:
  1. Split to large and small errors:

$$\mathbb{E}\left[\epsilon^2\right] \leqslant \mathbb{E}\left[\epsilon^2 \middle| |\epsilon| \leqslant \frac{1}{\beta}\right] + P\left(|\epsilon| > \frac{1}{\beta}\right) \mathbb{E}\left[\epsilon^2 \middle| |\epsilon| > \frac{1}{\beta}\right]$$
$$\triangleq D_S + P_L D_L$$

  2. Small errors: second moment of the argmax of a Wiener process with drift [Zehavi '84]
  3. Large errors: probability of error for orthogonal signals with unequal prior [Ziv and Zakai '69]

Analog PPM—Uniform Source

Proposition: Uniform Source Performance

$$D \leqslant D_S + D_L P_L$$

- $D_S \triangleq \frac{13/8}{(\beta \mathrm{ENR})^2} \cdot \{1 + o\,(1)\}$
- $D_L \triangleq \frac{1}{6}\left(1 + \frac{2}{\beta} + \frac{4}{\beta^2}\right)$
- $P_L \triangleq \frac{\beta\sqrt{\mathrm{ENR}}\mathrm{e}^{-\frac{\mathrm{ENR}}{2}}}{16\sqrt{\pi}} \cdot \{1 + o\,(1)\}$

Theorem: Upper Bound on Optimal Distortion

- Setting $\beta = (312\sqrt{\pi})^{\frac{1}{3}}\,(\mathrm{ENR})^{-\frac{5}{6}}\,\mathrm{e}^{\frac{\mathrm{ENR}}{6}}$ yields

$$D \leqslant 0.072\,\mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \mathrm{ENR}^{-\frac{1}{3}} \cdot \{1 + o(1)\}$$

Analog PPM—Gaussian Source

Proposition: Gaussian Source Performance

$$D \leqslant D_S + D_L$$

- $D_S \triangleq \frac{13/8}{(\beta \mathrm{ENR})^2} \cdot \{1 + o\,(1)\}$
- $D_L \triangleq 2\beta\sqrt{\mathrm{ENR}} \cdot \mathrm{e}^{-\frac{\mathrm{ENR}}{2}} \cdot \{1 + o\,(1)\}$

Theorem: Upper Bound on Optimal Distortion

- Setting $\beta = \left(\frac{13}{8}\right)^{\frac{1}{3}} (\mathrm{ENR})^{-\frac{5}{6}} \mathrm{e}^{\frac{\mathrm{ENR}}{6}}$ yields

$$D \leqslant 3 \cdot \left(\frac{13}{8}\right)^{\frac{1}{3}} \mathrm{e}^{-\frac{\mathrm{ENR}}{3}} \cdot \mathrm{ENR}^{-\frac{1}{3}} \cdot \{1 + o(1)\}$$

Numerical Evaluation

- Numerical optimization of the empirical performance and the complete bounds

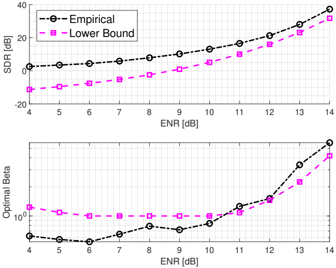


Figure: Gaussian source SDR

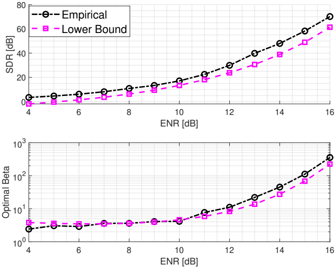


Figure: Uniform source SDR

- **Exponentially optimal, achieves next polynomial order**

Numerical Evaluation

- Comparison to Burnashev and Sevinç—Tuncel performance

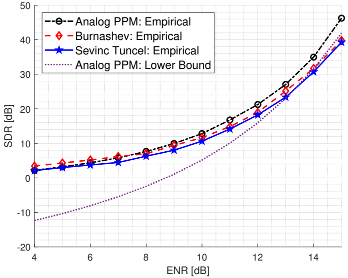


Figure: Gaussian source SDR

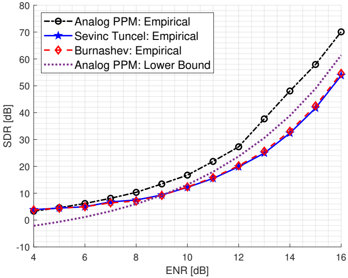


Figure: Uniform source SDR

Analog PPM—Robustness

"Why Analog PPM is good for Unknown ENR"

ENR Robustness

For $\mathrm{ENR} > \mathrm{ENR}_0$, the term $D_L P_L$ is negligible:

$$D \cong \frac{13/8}{(\beta \mathrm{ENR})^2}$$

- $\mathrm{ENR} > \mathrm{ENR}_0$ — **quadratic decay**

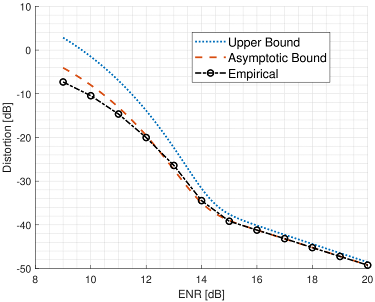


Figure: Gaussian source, $\beta = 3.68$

Known ENR Summary

Summary

- Separation based schemes:
  1. Does not achieve additional polynomial decay
  2. Does not improve for $\mathrm{ENR} > \mathrm{ENR}_0$
- We circumvent those drawbacks by using analog JSCC
  1. Achieves both exponential and *polynomial* decay terms—**Settles an open problem about the gap between Burnashev's bounds**
  2. Improves quadratically for $\mathrm{ENR} > \mathrm{ENR}_0$

Limitations

- β can be optimized only for specific ENR
- For constant β, the distortion decays quadratically

Part II

Unknown ENR

Unknown ENR Regime

Reminder

- Now, only the receiver knows the noise level N
- The transmitter accommodate for multiple noise levels
- **Main goal**: minimum energy to achieve

$$D \leqslant \sigma_x^2 \mathscr{F}(N), \forall N \geqslant 0$$

We concentrate on polynomial profiles:

$$\mathscr{F}(N) \triangleq \frac{1}{1 + \left(\frac{\tilde{E}}{N}\right)^L}, \tilde{E} \in \mathbb{R}^+$$

- Scenario received much attention in the finite bandwidth case [Mittal–Phamdo '02], [Reznic–Feder–Zamir '06], [Santhi–Vardy '06], [Bhattad–Narayanan '10]
- Results for the infinite bandwidth scenario attained recently [Köken–Tuncel '17], [Baniasadi–Tuncel '20]

Outer Bounds

Polynomially Decay Profile

- Separation based bound [Baniasadi–Tuncel '20]:

$$E_{\min}\left(L=2\right) \geqslant 0.804\tilde{E}$$

- Improved family of bounds: utilize the connection between robust transmission and the broadcast channel [Reznic–Feder–Zamir '06], [Köken–Tuncel '17], [Baniasadi–Tuncel '20]
- Best known bound [Baniasadi–Tuncel '20]:

$$E \geqslant E_{\min}\left(L=2\right) \cong 0.905\tilde{E}$$

Exponentially Decay Profile [Köken–Tuncel '17]

- Exponentially decay profile: $E_{\min} \to \infty$

Achievability: Hybrid Digital–Analog Schemes

Layered Transmission AWGN With $W \to \infty$ [Köken–Tuncel '17]

- Transmitter:
  - Uncoded transmission: $Y_0 = \sqrt{E_0} X + N_0$
  - Digital transmission: $S_k = Q\left(e_k\right), e_k = S_{k-1} - Q\left(S_{k-1}\right)$
- Receiver:
  1. Digital: Decodes $\{S_k\}_{k=1}^{\ell}$, ℓ is determined from noise level
  2. Analog: Estimates $\hat{e}_\ell$ from $\left\{\hat{S}_k\right\}_{k=1}^{\ell}$ and Y_0, to generate $\hat{X}$
- Total energy: $E_{\min} \cong 3.184\tilde{E}$

Multiple Uncoded Transmissions [Baniasadi–Tuncel '20]

- Combining multiple uncoded transmissions
- Minimal achievable energy: $E_{\min} \cong 2.32\tilde{E}$

Achievability: Hybrid Digital–Analog Schemes

Drawbacks

- Performance saturation:
  - Quantization does not improve with ENR
  - Uncoded transmissions—improve only *linearly*
- Linear transmission—inefficient use of BW

Finite BW: Modulo-lattice modulation (MLM) [Reznic–Feder–Zamir '06]

Modulo-based Wyner–Ziv coding outperforms success. refinement.

Our Approach: Two improvements

1. Use MLM in infinite BW setting:
   All layers designed for ENR $<$ True ENR, improve with ENR
2. Replace linear transmissions with analog PPM:
   Better utilization of BW

Modulo Lattice Modulation (MLM)

MLM

- Source comprises of known (j^k) and unknown (q^k) parts

$$x^k = q^k + j^k$$

Transmitted over an additive channel:

$$y^k = m^k\left(x^k\right) + z^k, \quad \frac{\frac{1}{k}\mathbb{E}\left[\left\|m^k\right\|^2\right]}{\sigma_z^2} \triangleq \mathrm{SNR}$$

- **Transmitter:**

$$m^k = [\eta x^k + d^k]_\Lambda$$

  - $d^k \sim \mathrm{Unif}\left(\mathscr{V}_0\right)$, dither signal

- **Receiver:**

$$\tilde{y}^k = [\alpha_c y^k - \eta j^k - d^k]_\Lambda = [\eta q^k + z^k_{\mathrm{eff}}]_\Lambda,$$

$$\hat{x}^k = \frac{\alpha_s}{\eta}\tilde{y}^k + j^k$$

Modulo-Lattice Modulation (MLM)

MLM Performance [Kochman Zamir '09, Ordentlich Erez '16]

Let q^k and z^k be *semi norm-ergodic* sequences. Then,

$$D \leqslant \frac{\sigma_q^2}{1+\mathrm{SNR}} \triangleq D^*$$

- Semi norm-ergodic:

$$\Pr\left(a^k \notin \mathscr{B}\left(0, \sqrt{(1+\delta)\, k\sigma_a}\right)\right) \leqslant \epsilon$$

Universality [Kochman–Zamir '09, Ordentlich–Erez '16]

- $\mathrm{SNR} \geqslant \mathrm{SNR}_0$, $\sigma_q^2 \leqslant \tilde{\sigma}_q^2$
- Tx oblivious of SNR and σ_q^2, knows SNR_0 and $\tilde{\sigma}_q^2$

$$D \leqslant \min\left\{\frac{\sigma_q^2}{1+\mathrm{SNR}_0}, \frac{\tilde{\sigma}_q^2}{1+\mathrm{SNR}} \cdot \frac{1+\mathrm{SNR}_0}{\mathrm{SNR}_0}\right\}$$

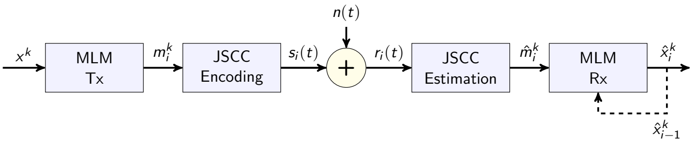
- For $\mathrm{SNR} \geqslant \mathrm{SNR}_0 \gg 1$: $D \rightarrow \frac{\sigma_q^2}{1+\mathrm{SNR}} = D^*$

M-Layer Transmitter

"Black-box" approach:

- **Generate set of "layers" using modulo-encoding (MLM)**
- **Transmit independently using scalar JSCC scheme**

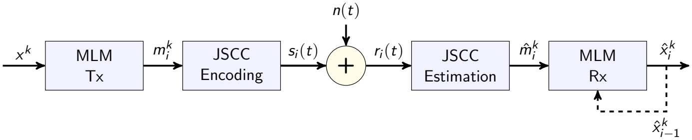
M Layer Transmitter:



Equivalent Channel

- MLM operates over the channel $m_i^k \rightarrow \hat{m}_i^k$
  1. Equivalent noise is JSCC output noise
  2. Noise not necessarily Gaussian

M-Layer Transmitter



M-Layer Transmitter

Transmitter:

- MLM part: Generates the set of signals

$$m_i^k = [\eta_i x^k + d_i^k]_\Lambda, \quad i = 1, \ldots, M$$

- JSCC part:
  1. Transmits the entries of x^k using linear transmission
  2. Transmits the entries of m_i^k using scalar JSCC scheme with E_i

Receiver: For $i = 1, \ldots, M$

- JSCC part: Generates $\hat{m}_i^k$ using the JSCC receiver
- MLM part: Uses the MLM receiver over the channel $m_i^k \to \hat{m}_i^k$ with $\hat{x}_{i-1}^k$ as SI, to generate $\hat{x}_i^k$

M-Layer Transmitter: Remarks

Interleaving

- Interleaving is used between MLM and JSCC steps
  - Receiver applies de-interleaving between JSCC and MLM
- Ensures independence between vector entries
- Allows to use MLM results for semi norm-ergodic variables

Gaussianization [AMIMON '07, No–Weissman '16, Erez–Hadad '16]

- PPM inputs multiplied by orthogonal matrix
- Allows to use analysis of JSCC with Gaussian inputs
- Receiver multiplies by $H^{-1} = H^T$—Gaussianizes the noise

- Dimension k is general—closed form solutions only for $k \to \infty$

M-Layer Transmitter: Linear

M-Layer Transmitter: Linear Transmissions

Let $L > 1, \tilde{E} > 0$ and $k \to \infty$. Then, polynomial profile is achievable for any transmit energy E that satisfies

$$E > \delta_{\text{lin}}(L)\, \tilde{E},$$

where

$$\delta_{\text{lin}}(L) \triangleq \frac{1}{2} \cdot \min_{(\alpha,x)\in\mathbb{R}_+^2} \left\{ \left(\frac{\mathrm{e}^{\alpha}}{x}\right)^{L-1} + \frac{x}{2}\left(\mathrm{e}^{\alpha L} - 1\right)\left(1 + \sqrt{1 + \frac{4\mathrm{e}^{\alpha(L+1)}}{\left(1-\mathrm{e}^{\alpha L}\right)^2}}\right)\frac{\mathrm{e}^{-2\alpha}}{1-\mathrm{e}^{-\alpha}} \right\}$$

- For $L = 2$, we get $E > 2.167\tilde{E}$
  1. **Better than** $2.32\tilde{E}$ **of other schemes**
  2. Achieved only by MLM (no PPM yet)

M-Layer Transmitter: PPM

M-Layer Transmitter: PPM Transmissions

Let $L = 2, \tilde{E} > 0$ and $k \to \infty$. Then, polynomial profile is achievable for any transmit energy E that satisfies

$$E > 1.961\tilde{E},$$

- Offers further improvement over linear-based transmitter
- PPM bounds are not tight—**improvement is even higher**

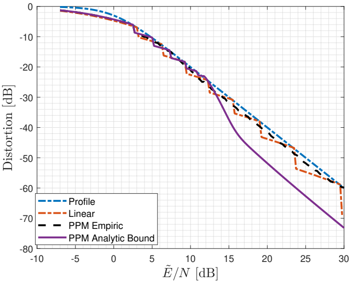


Figure: Distortion

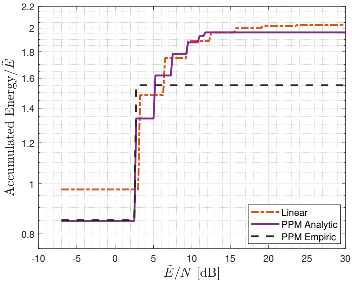


Figure: Accumulated Energy

M-Layer Transmitter: Scalar

- Numerical simulation of uniform source with $k = 1$
- Quadratic profile

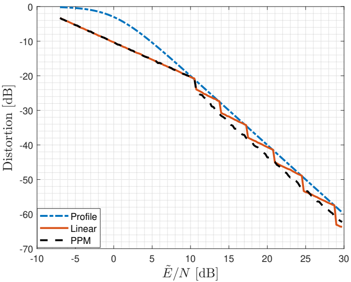


Figure: Distortion

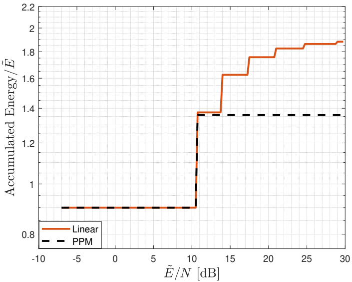


Figure: Accumulated Energy

- Scheme applies for practical low-delay scenario
  - **PPM allows saving of transmitted energy**

Unknown ENR Summary

Summary

- We introduced robust energy-efficient communication scheme
  1. Based on MLM and JSCC
  2. Analog PPM offers performance boost
- Total energy is lower than best reported results
  - Real improvement is even higher than theoretical bounds
- Scheme can be used in practical low-delay scenarios

Future Research

Known ENR

- Optimal polynomial decay:
  1. Burnashev's bound: $K_1 = K_2$
  2. MAP estimator is sub-optimal [Ibragimov–Khas'minskii '75]: Analysis of full MMSE decoder
- Vector sources: multidimensional mappings

Unknown ENR

- Tightening bounds:
  1. Analysis of inner-bound can be improved
  2. Different line of works: outer bound
- Universal SI at the receiver
- Dual problem: near-zero bandwidth [Baniasadi–Tuncel '20]

Questions?